CRYSTALLIZATIONS OF PL 4-MANIFOLDS

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Categories for n-dimensional manifolds

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DIFF category

- **smooth manifolds**, up to *diffeomorphisms*

Classification in dimension 3 and 4

n=3

- **TOP=PL** (any topological 3-manifold admits a PL-structure which is unique up to PL-isomorphisms)
- **PL=DIFF** (each PL-structure on a 3-manifold is smoothable in a unique way up to diffeomorphisms)

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- **PL=DIFF** (each PL-structure on a 4-manifold is smoothable and each PL-isomorphism is isotopic to a diffeomorphism)
- **TOP≠DIFF**
  - there are topological 4-manifolds admitting no smooth structure;
  - there can be non-diffeomorphic smooth structures on the same topological 4-manifold.

4-dimensional classical results

For **closed simply-connected oriented 4-manifolds** the well-known topological classification by Freedman is based on the **intersection form** defined on the second integral cohomology group of the manifold (modulo its torsion).

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[Freedman, 1982]

- for an **even** form λ there is exactly one homeomorphism class of simply connected closed manifolds having λ as intersection form;
- for an **odd** form λ there are exactly two classes (distinguished by the Kirby-Siebenmann invariant in $\mathbb{Z}_2$), at most one of which admits smooth representatives (smoothness requires vanishing invariant).

4-dimensional classical results

[Donaldson, 1983] [Furuta, 2001]

Closed simply-connected smooth 4-manifolds have intersection forms of the following types:

$$r[1] \oplus r'[-1] \qquad s\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \pm 2nE_8 \oplus t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ with } t > 2n.$$

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Up to now there is no classification of smooth structures on any given smoothable topological 4-manifold.

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Some recent results:

[Akhmedov-Doug Park, 2010], [Akhmedov-Ishida-Doug Park, 2013]

There exist (infinitely many) non-diffeomorphic smooth structures on:

- $\#_{2h-1}\mathbb{CP}^2\#_{2h}(-\mathbb{CP}^2)$, for any integer $h \geq 1$
- $\#_{2h-1}(\mathbb{S}^2 \times \mathbb{S}^2)$, for $h \geq 138$
- $\#_{2h-1}(\mathbb{CP}^2\#(-\mathbb{CP}^2))$, for $h \geq 23$
- $\#_{2p}(\mathbb{S}^2 \times \mathbb{S}^2)$ and $\#_{2p}(\mathbb{CP}^2\#(-\mathbb{CP}^2))$, for large enough integers p not divisible by 4.

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The existence of exotic PL-structures on $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2\#\mathbb{CP}^2$ or $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ is still an open problem!

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In dimension $n \geq 4$, where $TOP \not\cong PL$, a purely combinatorial approach to general PL-manifolds is useful if it yields:

- **combinatorial moves** which realize PL-homeomorphism (and not only TOP-homeomorphism);
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In particular:

- the *gem-complexity* $k(M^n)$ of a PL n-manifold M^n is the integer $p-1$, where $2p$ is the minimum order of a crystallization of M^n;
- the *regular genus* $\mathcal{G}(M^n)$ of an orientable (resp. non-orientable) PL n-manifold M^n is defined as the minimum genus (resp. half the minimum genus) of a surface into which a crystallization of M^n regularly embeds.

Regular genus and gem-complexity: lower bounds for $n = 4$

[Basak - Casali, 2015]

$$k(M^4) \ \geq \ 3\chi(M^4) + 10rk(M^4) - 6$$

$$\mathcal{G}(M^4) \ \geq \ 2\chi(M^4) + 5rk(M^4) - 4$$

where:
$\chi(M^4) =$ Euler characteristic of M^4 (closed PL 4-manifold)
$rk(M^4) =$ rank of the fundamental group $\pi_1(M^4)$ of M^4

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In the **simply-connected case**:

$$k(M^4) \ \geq \ 3\beta_2(M^4) \qquad \mathcal{G}(M^4) \ \geq \ 2\beta_2(M^4)$$

where $\beta_2(M^4)$ = second Betti number of M^4

TOP classification according to regular genus and gem-complexity for $n = 4$

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Hence, as a consequence of the up-to-date results about topological classification of simply connected PL 4-manifolds:

[Casali, 2012] [Casali - Cristofori, 2015]

Let M^4 be a simply-connected closed PL 4-manifold. If either $k(M^4) \leq 65$ or $\mathcal{G}(M^4) \leq 43$, then M^4 is TOP-homeomorphic to

$$(\#_r \mathbb{CP}^2)\#(\#_{r'}(-\mathbb{CP}^2)) \quad \textit{or} \quad \#_s(\mathbb{S}^2 \times \mathbb{S}^2),$$

where $r + r' = \beta_2(M^4)$, $s = \frac{1}{2}\beta_2(M^4)$

TOP classification according to regular genus and gem-complexity for $n = 4$

Sketch of the proof:
By [Donaldson, 1983], only forms of type

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The thesis easily follows from the fact that both $k(M^4) \leq 65$ and $\mathcal{G}(M^4) \leq 43$ imply $\beta_2 < 22$. □

General properties of the regular genus

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- For every $n \geq 2$,

$$\mathcal{G}(M^n) = 0 \iff M^n \cong \begin{cases} \mathbb{S}^n & \text{if } \partial M^n = \emptyset \\ \#_h \mathbb{D}^n & \text{if } \partial M^n \text{ has } h \text{ connected components} \end{cases}$$

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- For every n-manifold M^n $(n \geq 3)$, $\mathcal{G}(M^n)$ is a non-negative integer invariant, so that

$$\mathcal{G}(M^n) \geq \mathcal{G}(\partial M^n) \quad \text{and} \quad \mathcal{G}(M^n) \geq rk(M^n)$$

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Conjecture I_n

$\mathcal{G}(M_1^n \# M_2^n) = \mathcal{G}(M_1^n) + \mathcal{G}(M_2^n)$, for any M_1^n, M_2^n closed (orientable) PL n-manifolds.

The case of "low" regular genus

[Gagliardi, 1989] [Cavicchioli, 1989 - 1992] [Cavicchioli-Meschiari 1993]

- Let M^4 be an orientable 4-manifold, with $\partial M^4 = \emptyset$; then:

$$\mathcal{G}(M^4) = \rho \leq 3 \quad \Longrightarrow \quad M^4 \cong \begin{cases} \#_\rho(\mathbb{S}^3 \times \mathbb{S}^1) \\ \#_{\rho-2}(\mathbb{S}^3 \times \mathbb{S}^1) \# \mathbb{CP}^2 \end{cases}$$

- Let M^4 be a non-orientable 4-manifold, with $\partial M^4 = \emptyset$; then:

$$\mathcal{G}(M^4) = \rho \leq 2 \quad \Longrightarrow \quad M^4 \cong \#_\rho(\mathbb{S}^3 \tilde{\times} \mathbb{S}^1)$$

The case of "low" regular genus

[Casali-Malagoli, 1997]

Let M^4 be an (orientable or non-orientable) 4-manifold, with $\partial M^4 \neq \emptyset$; then:

$$\mathcal{G}(M^4) = \rho \leq 2 \quad \Longrightarrow \quad M^4 \cong \begin{cases} \#_{\rho - {}^{\partial}\rho}(\mathbb{S}^3 \tilde{\times} \mathbb{S}^1) \# \ {}^{(h)}\overset{|\backsim|}{\mathbb{Y}}{}^4_{{}^{\partial}\rho} \\ \mathbb{CP}^2 \# (\#_h \mathbb{D}^4) \end{cases}$$

where $0 \leq {}^{\partial}\rho = \mathcal{G}(\partial M^4) \leq \rho$, $h \geq 1$ is the number of boundary components and ${}^{(h)}\overset{|\backsim|}{\mathbb{Y}}{}^4_r$ denotes the connected sum of $h \geq 1$ orientable or non-orientable 4-dimensional handlebodies of genus $\alpha_i \geq 0$ $(i = 1, \dots, h)$, so that $\sum_{i=1}^{h} \alpha_i = r$..

The case of "restricted gap" between regular genus and boundary regular genus

[Casali 1992]

Let M^4 be an (orientable or non-orientable) 4-manifold, with $h \geq 1$ boundary components. If $0 \leq m \leq 1$, then:

$$\mathcal{G}(M^4) - \mathcal{G}(\partial M^4) = m \quad \Longrightarrow \quad M^4 \cong \#_m(\mathbb{S}^3 \tilde{\times} \mathbb{S}^1) \# \, {}^{(h)}\overset{|\backsim|}{\mathbb{Y}}{}^4_{\partial_\rho}.$$

The case of "restricted gap" between regular genus and rank of the fundamental group

[Casali, 1996] [Casali-Malagoli, 1997]

Let M^4 be an (orientable or non-orientable) 4-manifold.

- $\mathcal{G}(M^4) = rk(M^4) = \rho \iff M^4 \cong \begin{cases} \#_\rho(\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1) \\ \#_{\rho-\partial\rho}(\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1) \# \; {}^{(h)}\overset{|\backsim|}{\mathbb{Y}}{}^4_{\partial\rho} \end{cases}$
- $\mathcal{G}(M^4) \neq rk(M^4) \implies \mathcal{G}(M^4) - rk(M^4) \geq 2$
- $\mathcal{G}(M^4) - rk(M^4) = 2$ and $\pi_1(M^4) = *_m\mathbb{Z} \iff$

$$M^4 \cong \begin{cases} \#_m(\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1) \# \mathbb{CP}^2 & \text{if } \partial M^4 = \emptyset \\ \#_{m-\partial\rho}(\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1) \# \mathbb{CP}^2 \# \; {}^{(h)}\overset{|\backsim|}{\mathbb{H}}{}^4_{\partial\rho} & \text{if } \partial M^4 \neq \emptyset \end{cases}$$

- No M^4 exists with $\partial M^4 = \emptyset$, $\mathcal{G}(M^4) - rk(M^4) = 3$ and $\pi_1(M^4) = *_m\mathbb{Z}$.

Handle-decomposition of PL 4-manifolds

Every closed PL 4-manifold M^4 admits a **handle-decomposition**

$$M^4 = H^{(0)} \cup (H_1^{(1)} \cup \dots \cup H_{r_1}^{(1)}) \cup (H_1^{(2)} \cup \dots \cup H_{r_2}^{(2)}) \cup (H_1^{(3)} \cup \dots \cup H_{r_3}^{(3)}) \cup H^{(4)}$$

where $H^{(0)} = \mathbb{D}^4$ and each p-handle $H_i^{(p)} = \mathbb{D}^p \times \mathbb{D}^{4-p}$ $(1 \leq p \leq 4)$ is endowed with an an embedding (called *attaching map*)
$f_i^{(p)} : \partial\mathbb{D}^p \times \mathbb{D}^{4-p} \to \partial(H^{(0)} \cup \dots (H_1^{(p-1)} \cup \dots \cup H_{r_{p-1}}^{(p-1)}))$.

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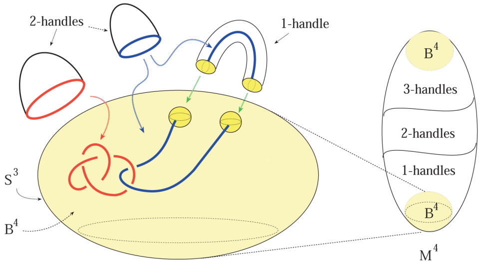

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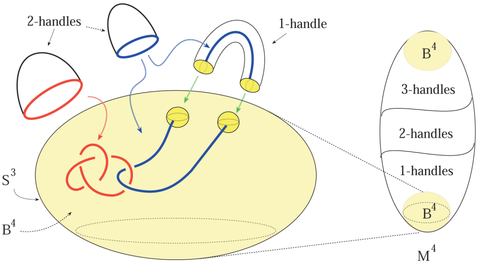

3- and 4-handles are uniquely attached to the union of 0, 1, 2-handles.

Handle-decomposition of PL 4-manifolds

If (Γ, γ) is a *crystallization* of a closed M^4 and $\{\{r,s,t\},\{i,j\}\}$ is a partition of the five vertices of the associated pseudocomplex $K(\Gamma)$, then M^4 admits a decomposition of type

$$M^4 \;=\; N(r,s,t) \cup_{\phi} N(i,j)$$

where:

- $N(r,s,t)$ denotes a regular neighborhood of the subcomplex of $K(\Gamma)$ generated by vertices labelled by $\{r,s,t\}$ *(union of 0,1,2-handles)*
- $N(i,j)$ denotes a regular neighborhood of the subcomplex of $K(\Gamma)$ generated by vertices labelled by $\{i,j\}$ *(union of 3,4-handles)*
- ϕ is a boundary identification.

Handle-decomposition of PL 4-manifolds

The hypotheses assumed about regular genus in many of the previous statements imply the associated handle-decomposition to *lack in 2-handles*; this fact allows to recognize the manifold M^4 as a connected sum of copies of $\mathbb{S}^3 \times \mathbb{S}^1$ and/or handlebodies.

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On the other hand, when at least a 2-handle appears, it is not possible to identify the represented 4-manifold, because of the great "freedom" in attaching 2-handles in dimension 4: *the attaching map for a 2-handle in dimension 4 depends on a framed knot* (K, c), with $c \in \mathbb{Z}$.

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However, if the union of 0,1,2-handles is known to have spherical boundary, then the attachment of a unique 2-handle is proved to give rise to a $\mathbb{CP}^2$ component, via an important result of Gordon-Luecke.

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In fact, if $\mathbb{S}^2 \times \mathbb{D}^2$ denotes the trivial $\mathbb{D}^2$-bundle over $\mathbb{S}^2$ and ξ_c, for every $c \in \mathbb{Z} - \{0, +1, -1\}$, denotes the non-trivial one with Euler class c and boundary $L(c, 1)$, then:

[Casali, 1996]

$$\mathcal{G}(\mathbb{S}^2 \times \mathbb{D}^2) = \mathcal{G}(\xi_c) = 3, \quad \forall c \in \mathbb{Z} - \{0, +1, -1\}.$$

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It is an open question *whether the number of PL 4-manifolds with fixed (possibly empty) boundary and fixed regular genus is finite or not*.

Simple and semi-simple crystallizations

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Definition ([Basak - Spreer, 2014]):

A crystallization (Γ, γ) of a closed PL 4-manifold M^4 is *simple* if $g_{ijk} = 1 \;\; \forall i, j, k \in \Delta_4$.
Equivalently: if any pair of vertices of $K(\Gamma)$ belongs to a unique 1-simplex (i.e. the 1-skeleton of $K(\Gamma)$ coincides with the 1-skeleton of a single 4-simplex).

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As a consequence: M^4 is simply-connected.

Definition ([Basak - Casali, 2015]):

A crystallization (Γ, γ) of a closed PL 4-manifold M^4 is *semi-simple* if $g_{ijk} = m + 1 \ \ \forall i, j, k \in \Delta_4$, where $m = rk(M^4)$.
Equivalently: any pair of vertices of $K(\Gamma)$ belongs to exactly $m + 1$ 1-simplices.

Characterization of manifolds admitting simple and semi-simple crystallizations

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[Basak - Casali, 2015]

A closed PL 4-manifold M^4 with $rk(M^4) = m$ admits semi-simple crystallizations if and only if $k(M^4) = 3\chi(M^4) + 10m - 6$.

If M^4 admits semi-simple crystallizations, then

$$\mathcal{G}(M^4) = 2\chi(M^4) + 5m - 4.$$

Manifolds admitting simple and semi-simple crystallizations

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$\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $K3$ and their connected sums

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[Basak - Spreer, 2014] [Basak - Casali, 2015]

Let M^4 and $M^{4\prime}$ be two PL 4-manifolds admitting simple (resp. semi-simple) crystallizations. Then, $M^4 \# M^{4\prime}$ admits simple (resp. semi-simple) crystallizations, too.

Examples of simple crystallizations

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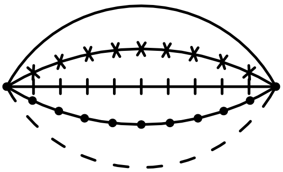

A simple crystallization of $\mathbb{S}^4$

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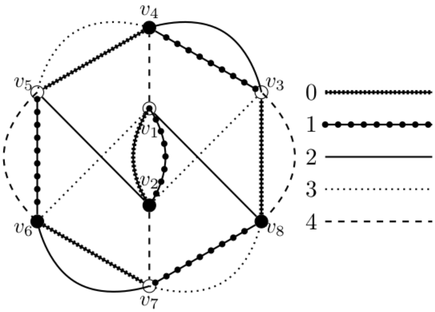
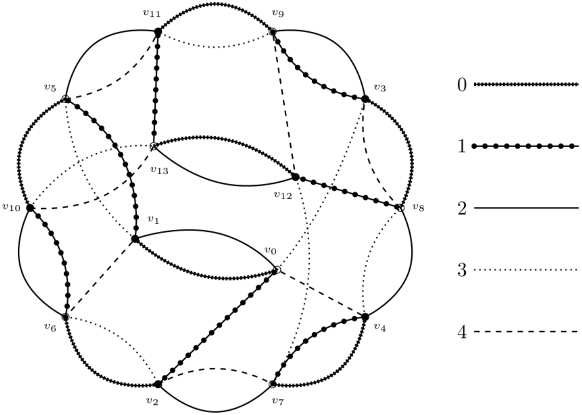


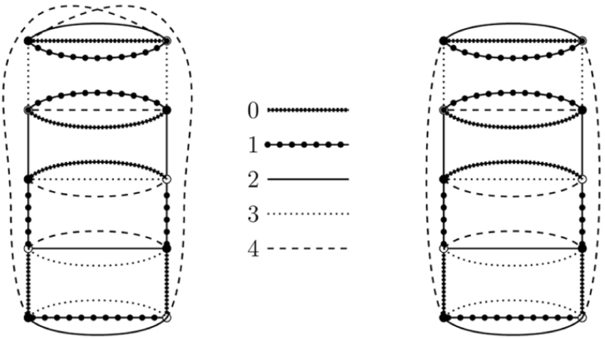
A simple crystallization of $\mathbb{CP}^2$

Examples of simple crystallizations



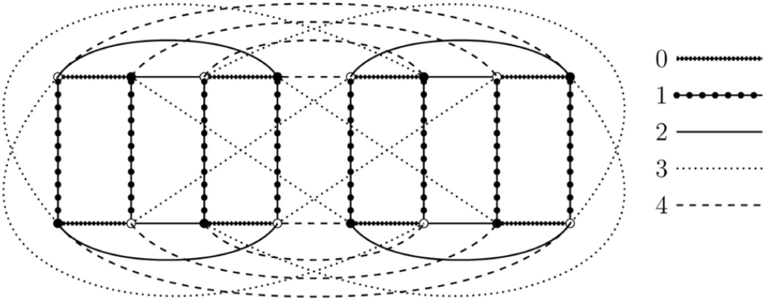
A simple crystallization of $\mathbb{S}^2 \times \mathbb{S}^2$

Examples of semi-simple crystallizations



Semi-simple crystallizations of $\mathbb{S}^3 \times \mathbb{S}^1$ and $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$

Examples of semi-simple crystallizations



A semi-simple crystallization of $\mathbb{RP}^4$

Simple and semi-simple crystallizations: additivity property

Simple and semi-simple crystallizations: additivity property

[Casali - Cristofori - Gagliardi, 2015] [Basak-Casali, 2015]

Let M^4 and $M^{4\prime}$ be PL 4-manifolds admitting simple or semi-simple crystallizations. Then:

$$\mathcal{G}(M^4 \# M^{4\prime}) = \mathcal{G}(M^4) + \mathcal{G}(M^{4\prime}) \quad \textit{and} \quad k(M^4 \# M^{4\prime}) = k(M^4) + k(M^{4\prime}).$$

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Consequence:

Let $M^4 \cong_{PL} (\#_p \mathbb{CP}^2)\#(\#_{p'}(-\mathbb{CP}^2))\#(\#_q(\mathbb{S}^2 \times \mathbb{S}^2))\#(\#_r(\mathbb{S}^3 \times \mathbb{S}^1))\# \#(\#_{r'}(\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1))\#(\#_s \mathbb{RP}^4)\#(\#_t K3)$, with $p, p', q, r, s, t \geq 0$. Then,

$$k(M) = 3(p + p' + 2q + 22t) + 4(r + r') + 7s$$

$$\mathcal{G}(M) = 2(p + p' + 2q + 22t) + r + r' + 3s.$$

In particular: $k(K3) = 66$ and $\mathcal{G}(K3) = 44$.

Handle decompositions/simple crystallizations

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[Casali - Cristofori - Gagliardi, 2015]

Let (Γ, γ) be a simple crystallization of a PL 4-manifold M^4. Then, for any partition $\{\{i,j,k\},\{r,s\}\}$ of Δ_4, the coloured triangulation $K(\Gamma)$ of M^4 induces a handle decomposition of M^4 consisting of one 0-handle, $\beta_2(M^4) = g_{rs} - 1$ 2-handles and one 4-handle.

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- $K(r,s)$ consists of exactly one 1-simplex (hence: $N(r,s) = \mathbb{D}^4$)
- $K(i,j,k)$ consists of g_{rs} 2-simplices, all having the same boundary (hence: $N(i,j,k) = \mathbb{D}^4 \cup (H_1^{(2)} \cup \cdots \cup H_{g_{rs}-1}^{(2)})$)

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Kirby Problem n. 50: *Does any closed simply-connected 4-manifold admit a handlebody decomposition without 1- and 3-handles?*

Exotic structures and simple crystallizations

[Casali - Cristofori, 2015]

Let M^4 and $M^{4\prime}$ be two closed PL 4-manifolds, with $M^4 \cong_{TOP} M^{4\prime}$. If both M^4 and $M^{4\prime}$ admit simple crystallizations, then $k(M^4) = k(M^{4\prime})$.

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Consequences:

- Let M^4 be $\mathbb{S}^4$ or $\mathbb{CP}^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \mathbb{CP}^2$ or $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$; if an exotic PL-structure on M^4 exists, then the corresponding PL-manifold does not admit simple crystallizations.
- If $r \in \{3, 5, 7, 9, 11, 13\} \cup \{r = 4n - 1 / n \geq 4\} \cup \{r = 4n - 2 / n \geq 23\}$, then infinitely many simply-connected PL 4-manifolds with $\beta_2 = r$ do not admit simple crystallizations.
- Let $\bar{M}$ be a PL 4-manifold TOP-homeomorphic but not PL-homeomorphic to $\mathbb{CP}^2 \#_2 (-\mathbb{CP}^2)$; then, either $\bar{M}$ does not admit simple crystallizations or $\bar{M}$ admits an order 20 simple crystallization.

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Note that, in case of positive answer, we would have the first example of a simple crystallization of an exotic PL 4-manifold.