REPRESENTING AND CLASSIFYING COMPACT PL 4-MANIFOLDS VIA REGULAR 5-COLORED GRAPHS

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special session "Geometric Topology, Manifolds, and Group Actions"

$(d+1)$-colored graphs

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Definition

A *$(d+1)$-colored graph* is (Γ, γ) where:

- $\Gamma = (V(\Gamma), E(\Gamma))$ regular multigraph of degree $d+1$,
- $\gamma : E(\Gamma) \to \Delta_d = \{0, \ldots, d\}$ such that $\gamma(e) \neq \gamma(f)$ for each pair of adjacent edges $e, f \in E(\Gamma)$ (*edge-coloration*)

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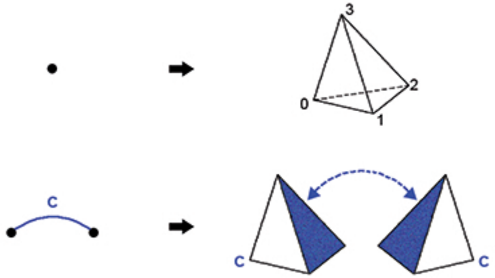
From any (Γ, γ) $\longrightarrow$ colored pseudocomplex $K(\Gamma)$

The pseudocomplex $K(\Gamma)$

1) take a d-simplex $\sigma(x)$ for every vertex $x \in V(\Gamma)$, and label its vertices by Δ_d;
2) if $x, y \in V(\Gamma)$ are joined by a c-colored edge, identify the $(d-1)$-faces of $\sigma(x)$ and $\sigma(y)$ opposite to c-labelled vertices, so that equally labelled vertices coincide.



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- ⋆ $K(\Gamma)$ is a *d-pseudomanifold*
- ⋆ (Γ, γ) *represents* $|K(\Gamma)|$
- ⋆ Γ is the 1-skeleton of the dual complex of $K(\Gamma)$

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  In particular:
- for each $c \in \Delta_d$, the c-labelled vertices of $K(\Gamma)$ are in bijection with the connected components of $\Gamma_{\hat{c}} = \Gamma_{\Delta_d - \{c\}}$ (representing $lk(v_c)$ in $K'(\Gamma)$).

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As a consequence:

$|K(\Gamma)|$ is a closed PL d-manifold if and only if, for every $c \in \Delta_d$, each connected component of $\Gamma_{\hat{c}}$ represents $\mathbb{S}^{d-1}$.

EXAMPLES:

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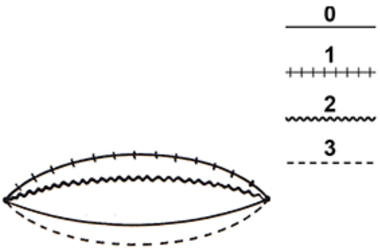

$\mathbb{S}^3$

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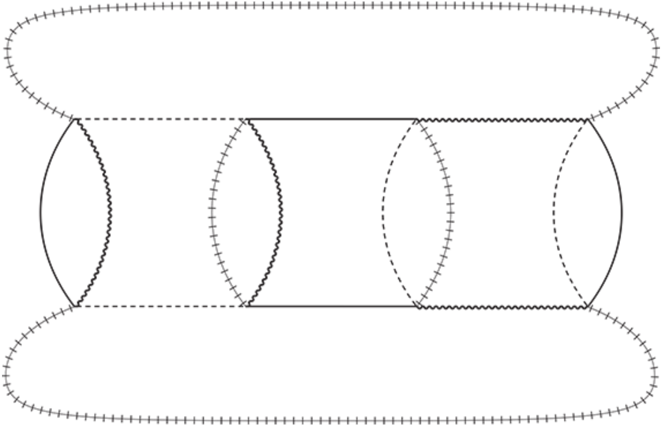

$\mathbb{S}^2 \times \mathbb{S}^1$

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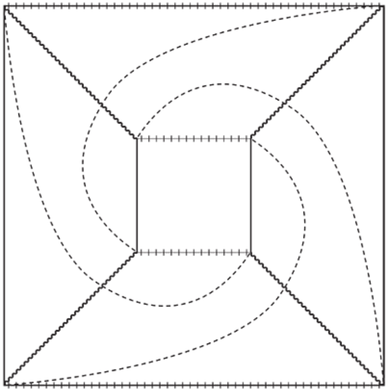

$L(2,1)$

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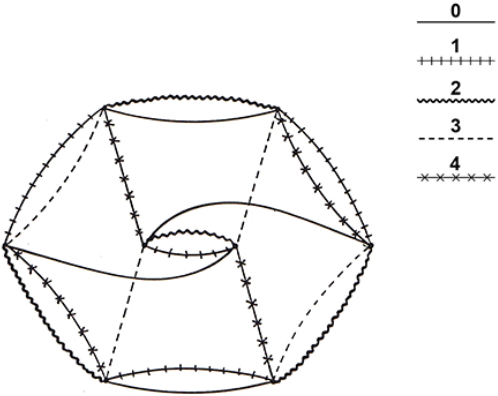

$\mathbb{CP}^2$

A recent approach to the boundary case

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Definition

A *singular d-manifold* is a compact connected d-dimensional polyhedron $|K|$ so that:

- for any vertex v, $lk(v)$ is a closed connected $(d-1)$-manifold;
- for any h-simplex σ, $h > 0$, $lk(\sigma) \cong \mathbb{S}^{d-h-1}$.

A vertex whose link is not a PL $(d-1)$-sphere is called *singular*.

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Note that singular d-manifolds include closed PL d-manifolds.

A recent approach to the boundary case

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Singular d-manifolds are in bijection with compact PL d-manifolds with no spherical boundary components.

A recent approach to the boundary case

If Γ is a regular $(d+1)$-colored graph:

$|K(\Gamma)|$ is a singular d-manifold if and only if, for every $c \in \Delta_d$, each connected component of $\Gamma_{\hat{c}}$ represents a closed $(d-1)$-manifold.

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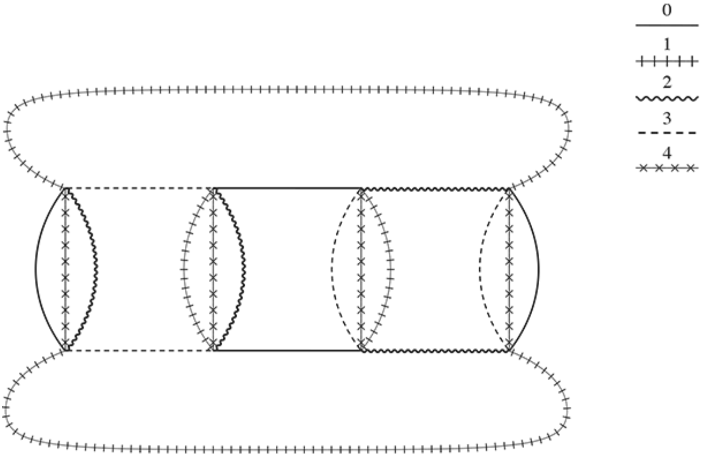
Existence theorem [Pezzana, 1974] [Casali-Cristofori-Grasselli, 2018]

Each singular d-manifold admits a regular $(d+1)$-colored graph representing it.

EXAMPLES: handlebodies



the orientable genus one 4-dimensional handlebody $\mathbb{Y}^4_1 = \mathbb{S}^1 \times \mathbb{D}^3$

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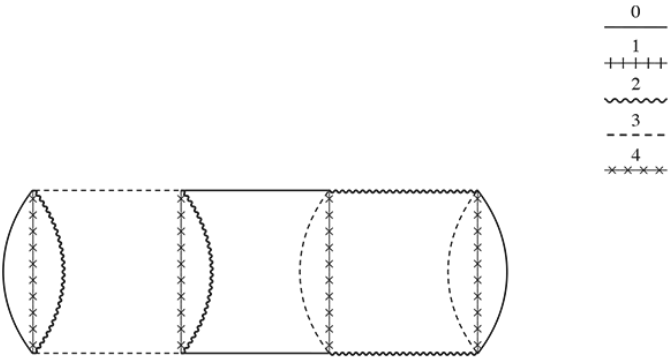

without color 1: $\mathbb{S}^3$

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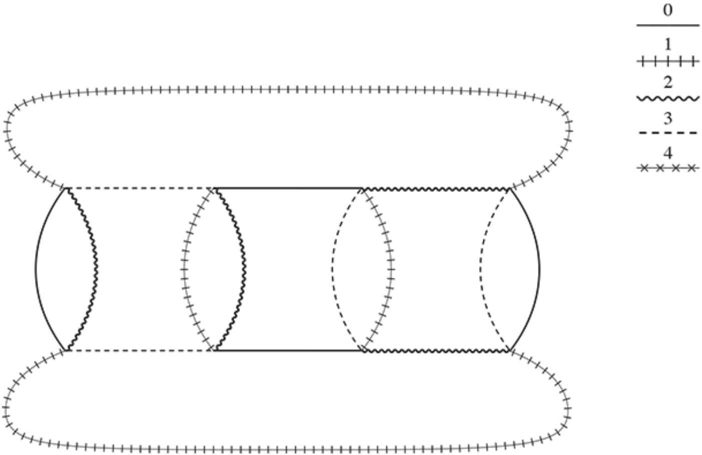

without color 4: $\mathbb{S}^2 \times \mathbb{S}^1 = \partial \mathbb{Y}^4_1$

CONSTRUCTIONS: (a) products $M \times I$

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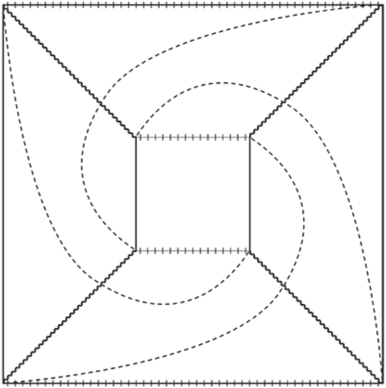

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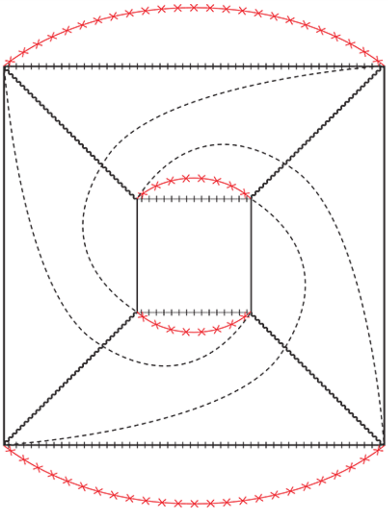

$L(2,1) \times I$

CONSTRUCTIONS: (b) 4-manifolds $M^4(L, c)$

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Let (L, c) be a framed link.
If L has l components $L_1, \dots, L_l$ and $c = (c_1, \dots, c_l)$,

$$M^4(L, c) \; = \; \mathbb{D}^4 \cup (H_1^{(2)} \cup \dots \cup H_l^{(2)})$$

where, $\forall i = 1, \dots, l$,

$$H_i^{(2)} = \mathbb{D}^2 \times \mathbb{D}^2$$

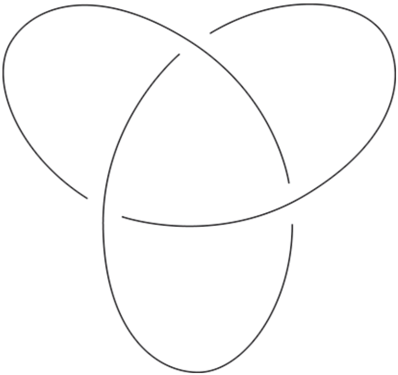
has attaching map

$$\phi_i : \partial\mathbb{D}^2 \times \mathbb{D}^2 \to \partial\mathbb{D}^4$$

such that $\phi_i(\mathbb{S}^1 \times \{0\}) = L_i$ has linking number c_i with $\phi_i(\mathbb{S}^1 \times \{x\})$, $\forall x \in \mathbb{D}^2 - \{0\}$.

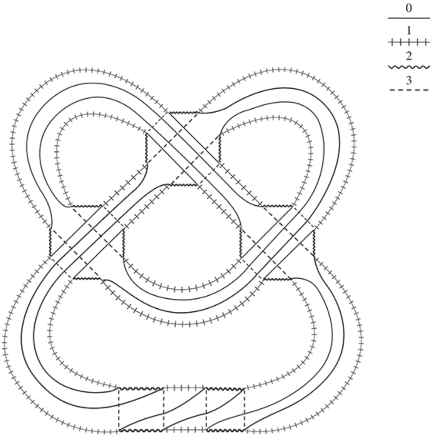
$\partial M^4(L, c) \; = \; M^3(L, c)$, obtained by Dehn surgery on (L, c).

CONSTRUCTIONS: (b) 4-manifolds $M^4(L, c)$



framed knot $(T, 1)$

CONSTRUCTIONS: (b) 4-manifolds $M^4(L, c)$



$M^3(T, 1)$, obtained by Dehn surgery on $(T, 1)$,
i.e. the Heegaard genus two Poincaré homology sphere

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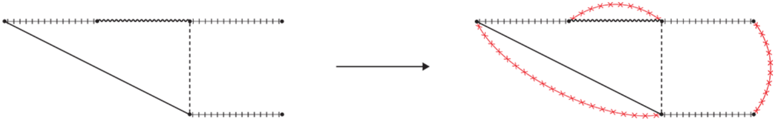

main step toward $M^4(L, c)$

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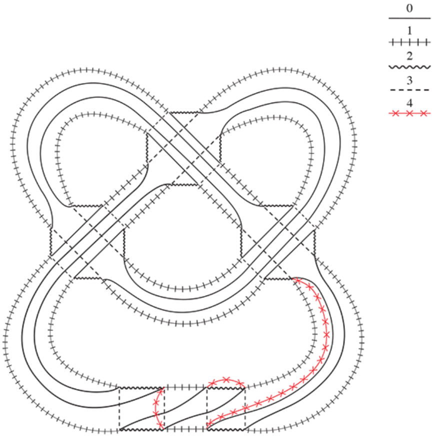

main step toward $M^4(T, 1)$, the 4-manifold with boundary associated to $(T, 1)$ (with boundary $M^3(T, 1)$)

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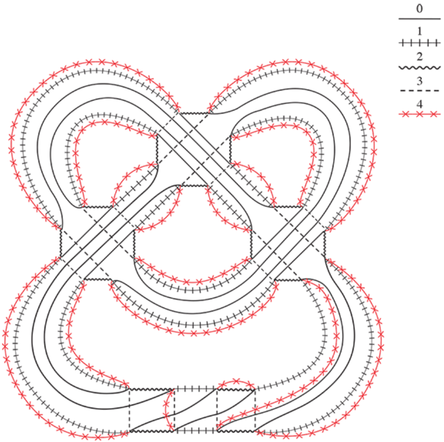

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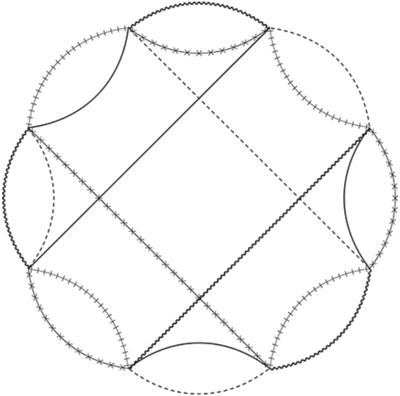
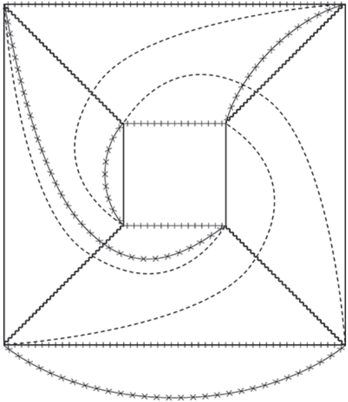


$M^4(K_0, 0) = \mathbb{S}^2 \times \mathbb{D}^2$

(with boundary $M^3(K_0, 0) = \mathbb{S}^2 \times \mathbb{S}^1$)

EXAMPLES: $\mathbb{D}^2$-bundles over $\mathbb{S}^2$



$M^4(K_0, 2) = \xi_2$, the $\mathbb{D}^2$-bundle over $\mathbb{S}^2$ with Euler class 2
(with boundary $M^3(K_0, 2) = L(2, 1)$)

Regular embeddings

A cellular embedding $i : |\Gamma| \to F$ of a $(d+1)$-colored graph (Γ, γ) into a (closed) surface F is called a regular embedding if there exists a cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_d)$ of Δ_d s.t. each connected component of $F - i(|\Gamma|)$ is an open ball bounded by the image of an $\{\varepsilon_i, \varepsilon_{i+1}\}-$ colored cycle of Γ.

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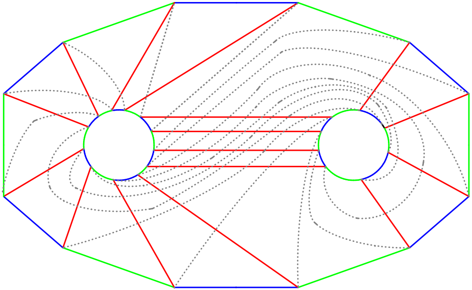
EXAMPLE: Regular embedding corresponding to $\varepsilon =$ (green, red, blue, grey)



The regular genus

Theorem [Gagliardi, 1981]

For each $(d+1)$-colored graph (Γ, γ) and for every cyclic permutation ε of Δ_d, there exists a *regular embedding* of Γ onto a suitable surface F_ε. Moreover:

- F_ε is orientable if and only if Γ is bipartite;
- ε and ε^{-1} induce the same embedding.

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Definition

The *regular genus* $\rho_\epsilon(\Gamma)$ of Γ with respect to ε is the classical genus (resp. half of the genus) of the orientable (resp. non-orientable) surface F_ε :

$$\sum_{i\in\mathbb{Z}_{d+1}} g_{\varepsilon_i\varepsilon_{i+1}} + (1-d)p = 2 - 2\rho_\varepsilon(\Gamma)$$

where $g_{\epsilon_i,\epsilon_{i+1}}$ = number of $\{\epsilon_i, \epsilon_{i+1}\}$-cycles, $2p = \#V(\Gamma)$.

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The regular genus is a PL-manifold invariant which extends to arbitrary dimension the classical genus of a surface and the Heegaard genus of a 3-manifold.

The generalized regular genus

Definition

The *generalized regular genus* of a singular d-manifold N is defined as

$$\overline{\mathcal{G}}(N) = \min\{\rho_\epsilon(\Gamma) \mid (\Gamma, \gamma) \text{ represents } N, \quad \epsilon \text{ cyclic permutation of } \Delta_d\}.$$

The *generalized regular genus* of the associated compact d-manifold $\check{N}$ is

$$\overline{\mathcal{G}}(\check{N}) = \overline{\mathcal{G}}(N).$$

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- for any singular d-manifold N with one singular vertex,

$$\bar{\mathcal{G}}(N) \geq \mathcal{G}(\partial \check{N}).$$

Classifying via generalized regular genus

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Theorem [Gagliardi, 1989] [Cavicchioli, 1989]

Let M^4 be a closed PL 4-manifold. Then,

$$\mathcal{G}(M^4) = 1 \iff M^4 \in \{\mathbb{S}^1 \times \mathbb{S}^3,\ \mathbb{S}^1 \tilde{\times} \mathbb{S}^3\}$$

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Theorem [Casali, preprint 2018]

Let N^4 be a singular 4-manifold with $\bar{\mathcal{G}}(N^4) = 1$. Then,

either $N^4 \in \{\mathbb{S}^1 \times \mathbb{S}^3, \ \mathbb{S}^1 \tilde{\times} \mathbb{S}^3\}$ or $\check{N}^4 \in \{\mathbb{Y}^4_1, \tilde{\mathbb{Y}}^4_1\}$ or $\check{N}^4 \cong \bar{M} \times I$,

where $\bar{M}$ is a genus one closed 3-manifold.

Classifying via generalized regular genus

Theorem [Cavicchioli, 1992]

Let M^4 be a closed PL 4-manifold. Then,

$$\mathcal{G}(M^4) = 2 \iff M^4 \in \{\#_2(\mathbb{S}^1 \times \mathbb{S}^3),\ \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3),\ \mathbb{CP}^2\}$$

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Theorem [Casali, preprint 2018]

Let N^4 be a singular 4-manifold with one singular vertex at most and with $\bar{\mathcal{G}}(N^4) = 2$. Then:

$$\text{either} \quad N^4 \in \{\#_2(\mathbb{S}^1 \times \mathbb{S}^3),\ \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3),\ \mathbb{CP}^2\},$$

$$\text{or} \quad \check{N}^4 \in \{\mathbb{Y}^4_2,\ \tilde{\mathbb{Y}}^4_2,\ \mathbb{Y}^4_1\#(\mathbb{S}^1 \times \mathbb{S}^3),\ \tilde{\mathbb{Y}}^4_1\#(\mathbb{S}^1 \times \mathbb{S}^3),\ \mathbb{S}^2 \times \mathbb{D}^2,\ \xi_2\},$$

or $\check{N}^4 \cong M^4(K,d)$, (K,d) framed knot s.t. $M^3(K,d) = L(\alpha,\beta)$, $\alpha \geq 3$.

Classifying via generalized regular genus

Theorem [Casali, preprint 2018]

Let $\xi_c = M^4(K_0, c)$ be the $\mathbb{D}^2$-bundle over $\mathbb{S}^2$ with Euler class c, $\forall c \in \mathbb{Z}^+ - \{1\}$ (whose boundary is $M^3(K_0, c) \cong L(c, 1)$).

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Generalized regular genus is NOT finite-to-one within the class of compact manifolds.

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OPEN QUESTION:
is the number of PL 4-manifolds with fixed (possibly empty) boundary and fixed generalized regular genus finite or not?

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$$\bar{\mathcal{G}}(\xi_c) = \bar{\mathcal{G}}(\mathbb{S}^2 \times \mathbb{D}^2) = 2.$$

OPEN QUESTION:
Does a framed knot (K, d) exist, with non-trivial K, so that $\bar{\mathcal{G}}(M^4(K, d)) = 2$ (and $M^3(K, d) = L(\alpha, \beta)$, with $\alpha \geq 3$) ?

Classifying via generalized regular genus

Theorem [Casali, preprint 2018]

Let N^4 be a singular 4-manifold with exactly one singular vertex. Then:

$$\bar{\mathcal{G}}(N^4) = \mathcal{G}(\partial \check{N}^4) = m \geq 1 \quad \Longleftrightarrow \quad \check{N}^4 \in \{\mathbb{Y}^4_m, \tilde{\mathbb{Y}}^4_m\}$$

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Let N^4 be a singular 4-manifold with one singular vertex at most. Then:

$$\bar{\mathcal{G}}(N^4) = rk(\pi_1(\check{N}^4)) = \rho \iff \text{either } N^4 \in \{\#_\rho(\mathbb{S}^1 \times \mathbb{S}^3), \#_\rho(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\}$$
$$\text{or } \check{N}^4 \in \{\#_{\rho-\partial_\rho}(\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{Y}^4_{\partial_\rho}, \#_{\rho-\partial_\rho}(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\tilde{\mathbb{Y}}^4_{\partial_\rho}\}$$

$$\bar{\mathcal{G}}(N^4) \neq rk(\pi_1(\check{N}^4)) \implies \bar{\mathcal{G}}(N^4) - rk(\pi_1(\check{N}^4)) \geq 2$$

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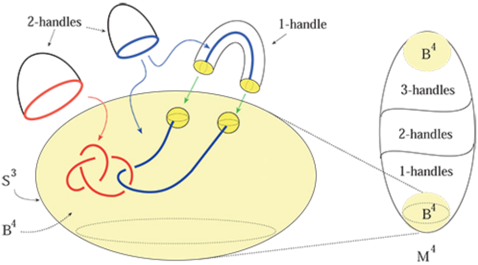
If (Γ, γ) is such a graph representing M^4 and $\{r,s,t\} \cup \{i,j\} = \Delta_5$:

$$M^4 = \underbrace{\mathbb{D}^4 \cup (H_1^{(1)} \cup \cdots \cup H_{r_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{r_2}^{(2)})}_{N(r,s,t)} \cup \underbrace{(H_1^{(3)} \cup \cdots \cup H_{r_3}^{(3)}) \cup H^{(4)}}_{N(i,j)}$$



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where:

- $N(r,s,t)$ = regular neighborhood of the subcomplex of $K(\Gamma)$ generated by vertices labelled by $\{r,s,t\}$ *(union of 0,1,2-handles)*
- $N(i,j)$ = regular neighborhood of the subcomplex of $K(\Gamma)$ generated by vertices labelled by $\{i,j\}$ *(union of 3,4-handles)*

HINT OF PROOFS:

- no 2-handles $\Rightarrow$ $M^4 \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^3)$
  (via [Montesinos, 1979] and [Laudenbach-Poenaru, 1972])

HINT OF PROOFS:

- no 2-handles $\Rightarrow$ $M^4 \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^3)$
  (via [Montesinos, 1979] and [Laudenbach-Poenaru, 1972])
- one 2-handle and $\partial N(r,s,t) \cong \mathbb{S}^2 \times \mathbb{S}^1$ $\Rightarrow$ $\check{N}^4 \cong \mathbb{S}^2 \times \mathbb{D}^2$
  (via [Gabai, 1987], ensuring $(K,d) = (K_0, 0)$)
- one 2-handle and $\partial N(r,s,t) \cong \mathbb{S}^3$ $\Rightarrow$ $M^4 \cong \mathbb{CP}^2$
  (via [Gordon-Luecke, 1989], ensuring $(K,d) = (K_0, 1)$)
- one 2-handle and $\partial N(r,s,t) \cong L(2,1)$ $\Rightarrow$ $\check{N}^4 \cong \xi_2$
  (via [Kronheimer-Mrowka-Ozsvath-Szabo, 2007], ensuring $(K,d) = (K_0, 2)$)

G-degree

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Definition

- *G-degree of a $(d+1)$-colored graph Γ:*

$$\omega_G(\Gamma) = \sum_{i=1}^{\frac{d!}{2}} \rho_{\varepsilon^{(i)}}(\Gamma)$$

with $\{\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(\frac{d!}{2})}\}$ set of all cyclic permutations of Δ_d (up to inverse), and $\rho_{\varepsilon^{(i)}}$ regular genus of Γ with respect to $\varepsilon^{(i)}$.

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- *G-degree of a singular d-manifold N:*

$$\mathcal{D}_G(N) = \min\{\omega_G(\Gamma) \mid (\Gamma, \gamma) \text{ represents } N\}.$$

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A $(d+1)-$*dimensional Colored Tensor Model* is a *formal partition function*

$$\mathcal{Z}[N, \{t_B\}_{B \in \mathcal{CG}(d)}] := \int_f \frac{dT d\overline{T}}{(2\pi)^{N^d}} e^{-N^{d-1}\overline{T} \cdot T + \sum_B \alpha_B B(\overline{T}, T)}$$

where:

- $T, \overline{T} \in \otimes_d \mathbb{C}^N$
- $\mathcal{CG}(d)$ is the set of bipartite d-colored graphs
- $B(\overline{T}, T)$ invariant encoded by a d-colored graph $B \in \mathcal{CG}(d)$, α_B depending on t_B

Theorem [Bonzom-Gurau, 2012]

If $\alpha_B = N^{d-1-\frac{2}{(d-2)!}\omega_G(B)} \frac{t_B}{|\mathrm{Aut}(B)|}$,

the *free energy* $\frac{1}{N^d} \log \mathcal{Z}[N, \{t_B\}]$ admits $\frac{1}{N}$ expansion

$$\sum_{\omega_G \geq 0} N^{-\frac{2}{(d-1)!}\omega_G} F_{\omega_G}[\{t_B\}] \in \mathbb{C}[[N^{-1}, \{t_B\}]]$$

where the coefficients $F_{\omega_G}[\{t_B\}]$ are generating functions of connected $(d+1)$-colored graphs with fixed G-degree ω_G.

Classifying via G-degree

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- For any closed PL 4-manifold M,

$$\mathcal{D}_G(M) = 6(\underbrace{k(M)}_{\text{PL}} + \underbrace{\chi(M)}_{\text{TOP}} - 2)$$

Classifying via G-degree

Theorem [Casali, preprint 2018]

Let N^4 be a singular 4-manifold with $\mathcal{D}_G(N^4) \leq 18$. Then:

- $\mathcal{D}_G(N^4) = 0 \iff N^4 \cong \mathbb{S}^4$;
- $\mathcal{D}_G(N^4) = 12 \iff$ either $N^4 \in \{\mathbb{S}^1 \times \mathbb{S}^3,\ \mathbb{S}^1 \tilde{\times} \mathbb{S}^3\}$ or $\check{N}^4 \in \{\mathbb{Y}^4_1,\ \tilde{\mathbb{Y}}^4_1\}$;
- $\mathcal{D}_G(N^4) = 18 \iff \check{N}^4 \in \{L(2,1) \times I,\ (\mathbb{S}^1 \times \mathbb{S}^2) \times I,\ (\mathbb{S}^1 \tilde{\times} \mathbb{S}^2) \times I\}$.

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Theorem [Casali, preprint 2018]

If N^4 is a singular 4-manifold with one singular vertex at most, then:

$$\mathcal{D}_G(N^4) = 24 \iff \text{either } N^4 \in \{\#_2(\mathbb{S}^1 \times \mathbb{S}^3),\ \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3),\ \mathbb{CP}^2\}$$
$$\text{or } \check{N}^4 \in \{\mathbb{Y}^4_2,\ \tilde{\mathbb{Y}}^4_2,\ \mathbb{Y}^4_1 \# (\mathbb{S}^1 \times \mathbb{S}^3), \tilde{\mathbb{Y}}^4_1 \# (\mathbb{S}^1 \times \mathbb{S}^3),\ \mathbb{S}^2 \times \mathbb{D}^2,\ \xi_2\}.$$

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where the coefficients $F_{\omega_G}[\{t_B\}]$ are generating functions of connected $(d+1)$-colored graphs with fixed G-degree ω_G.

For $d = 4$, all compact 4-manifolds involved in the starting terms of the $1/N$ expansion (up to $\omega_G = 24$) are now identified!