CATALOGUES OF PL-MANIFOLDS AND COMPLEXITY ESTIMATIONS VIA CRYSTALLIZATION THEORY

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Workshop “TRIANGULATIONS” - Oberwolfach - May 4, 2012

Coloured triangulations and coloured graphs

A *coloured triangulation* of a compact PL n-manifold M^n is a pair $(\bar{K}, \xi)$, where $\bar{K}$ is a pseudocomplex[1] triangulating M^n and $\xi : S_0(\bar{K}) \to \Delta_n = \{0, 1, \dots, n\}$ (*vertex-labelling*) satisfies:

i) each n-simplex of $\bar{K}$ has exactly one c-labelled vertex, for every $c \in \Delta_n$;

ii) each n-labelled vertex is internal in $\bar{K}$.

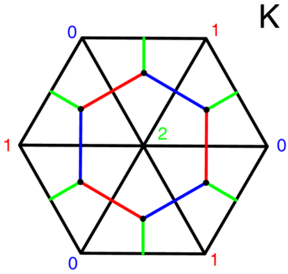
EXAMPLE: If $M^n = |K|$, then (K', ξ) is a coloured triangulation, where

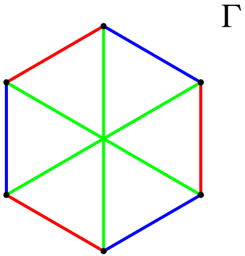
$$K' \text{ first barycentric subdivision of } K$$
$$\xi(v) = r \text{ iff } v \text{ barycenter of } \tau^r \in K$$

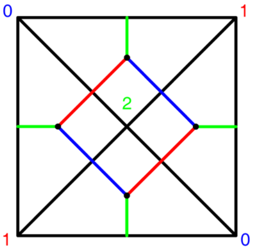
[1] This means that its "simplices" may intersect in more than one face.

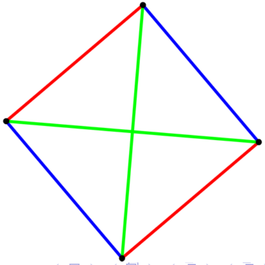
A coloured triangulation $\bar{K}$ of M^n is combinatorially visualized by means of an *(n+1)-coloured graph* (Γ, γ):

- $\Gamma = (V(\Gamma), E(\Gamma))$ is the 1-skeleton of the dual cellular complex of $\bar{K}$;
- $\gamma : E(\Gamma) \rightarrow \Delta_n$ (edge-coloration) is defined by: $\gamma(e) = c$ if $e \in E(\Gamma)$ is dual to an $(n-1)$-simplex of $\bar{K}$ having no c-labelled vertex.


K
0
1
1
2
0
0
1


Γ

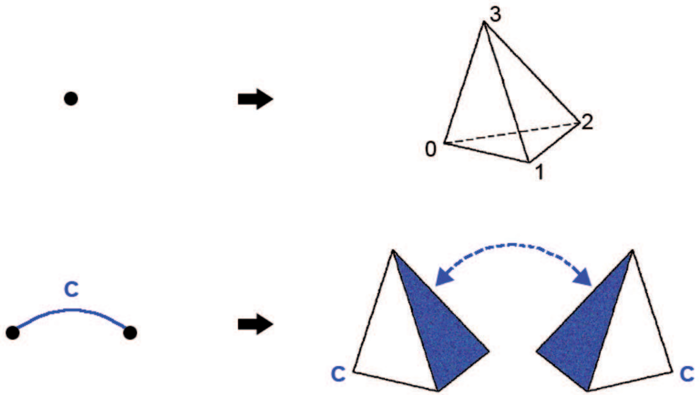

0
1
2
1
0



(Γ, γ) is said to *represent* M^n, since the reversed process allows to completely reconstruct the coloured triangulation $\bar{K} = K(\Gamma)$ - and hence $M^n = |K(\Gamma)|$ - from it:

1) take an n-simplex $\sigma(x)$ for every vertex $x \in V(\Gamma)$, and label its vertices by Δ_n;
2) if $x, y \in V(\Gamma)$ are joined by a c-coloured edge, identify the $(n-1)$-faces of $\sigma(x)$ and $\sigma(y)$ opposite to c-labelled vertices, so that equally labelled vertices coincide.

(Γ, γ) is also said to be a *gem* ("graph encoded manifold") of M^n.


3
2
0
1
c
c
c

CONSEQUENCES:

- If M^n is a closed manifold, any $(n+1)$-coloured graph representing it is a regular graph of degree $n+1$;
  If $\partial M^n \neq \emptyset$, any $(n+1)$-coloured graph representing M^n has a subset of vertices (*boundary vertices*) of degree n, lacking in n-coloured edges and corresponding to boundary n-simplices of $K(\Gamma)$.
- $M^n = |K(\Gamma)|$ is orientable iff Γ is bipartite;
  " " non-orientable " " " non-bipartite.
- $\forall \mathcal{B} \subset \Delta_n$, with $\#\mathcal{B} = h$, there is a bijection between $(n-h)$-simplices of $K(\Gamma)$ whose vertices are labelled by $\Delta_n - \{\mathcal{B}\}$ and connected components of h-coloured graph $\Gamma_{\mathcal{B}} = (V(\Gamma), \gamma^{-1}(\mathcal{B}))$.

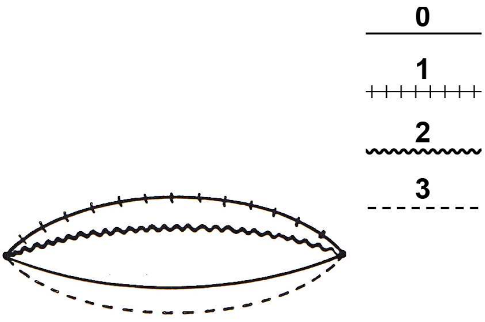
A *crystallization* of an n-manifold M^n is any $(n+1)$-coloured graph (Γ, γ) representing it, so that $K(\Gamma)$ has the minimal number of vertices.

If ∂M^n is either empty or connected, that minimal number is always equal to $n+1$:

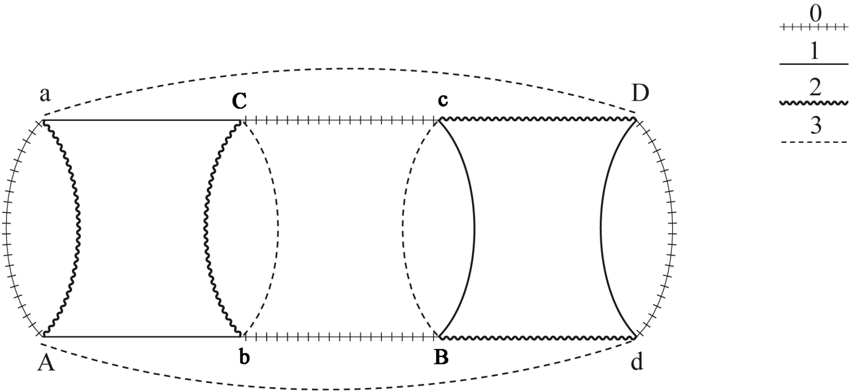
> *(Γ, γ) is a crystallization of $M^n = |K(\Gamma)|$ if and only if $\Gamma_{\hat{c}}$ is connected, $\forall c \in \Delta_n$*
> *(or, equivalently, if and only if $K(\Gamma)$ has exactly one c-labelled vertex, $\forall c \in \Delta_n$)*

Pezzana Existence Theorem (1974)

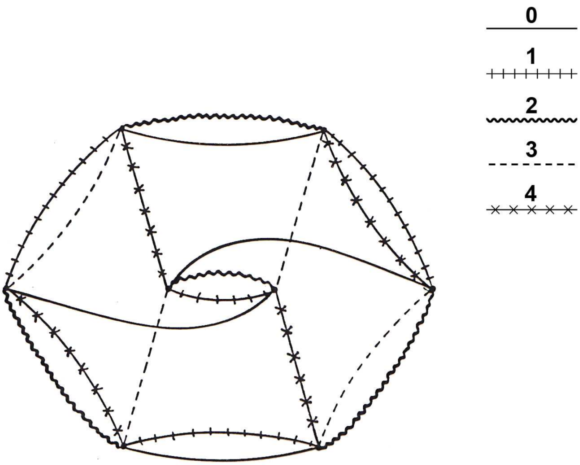
Each PL n-manifold M^n (with or without boundary) admits a crystallization.



$\mathbb{S}^3$



$\mathbb{S}^1 \times \mathbb{S}^2$



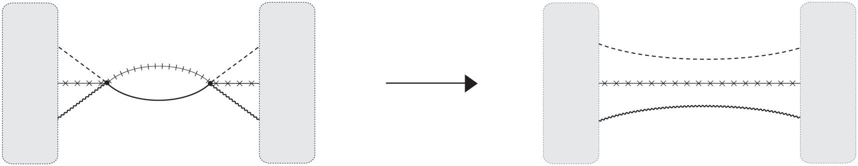
$\mathbb{CP}^2$

A complete (finite) set of graph-moves allows to translate the (PL)-homeomorphism problem for n-manifolds into an equivalence problem for $(n+1)$-coloured graphs:

two coloured graphs represent the same PL-manifold if and only if they can be obtained one each other by a finite sequence of dipole moves.

An *h-dipole* $(1 \leq h \leq n)$ of (Γ, γ) is a subgraph $\Theta = \{v, w\}$ consisting of two vertices $v, w \in V(\Gamma)$ joined by h edges coloured by $j_1, j_2, \dots, j_h \in \Delta_n$, such that:

- v and w belong to different components, Ξ_1 and Ξ_2 say, of $\Gamma_{\Delta_n - \{j_1,\dots,j_h\}} = (V(\Gamma), \gamma^{-1}(\Delta_n - \{j_1, \dots, j_h\}))$;
- if either v or w is an internal vertex, then either Ξ_1 or Ξ_2 is a regular graph of degree $n+1-h$.



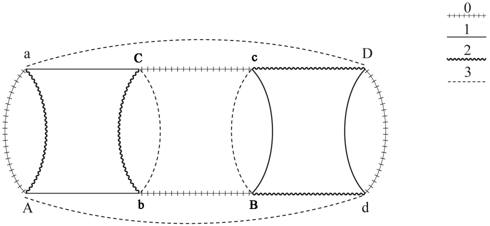
dipole move

2. Cataloguing PL-manifolds via crystallization theory

Each order $2p$ $(n+1)$-coloured graph (Γ, γ) (with $V(\Gamma) = \{v_1, \ldots, v_{2p}\}$) may be obviously encoded by an "incidence matrix"

$$A_\Gamma : \mathbb{N}_{2p} \times \mathbb{N}_{n+1} \to \{0, 1, \ldots, 2p\},$$

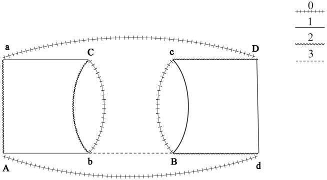
$$A_\Gamma(i, c) = \begin{cases} j & \text{if } v_i \text{ is } c\text{-adjacent to } v_j \\ 0 & \text{if } v_i \text{ has no } c\text{-adjacent vertex} \end{cases}$$



The standard crystallization of $\mathbb{S}^2 \times \mathbb{S}^1$

	colour 0	colour 1	colour 2	colour 3
vertex a	A	C	A	D
vertex A	a	b	a	d
vertex b	B	A	C	C
vertex B	b	c	d	c
vertex c	C	B	D	B
vertex C	c	a	b	b
vertex d	D	D	B	A
vertex D	d	d	c	a

If we consider a manifold with boundary, zero elements appear, corresponding to boundary vertices of the graph.



A crystallization of $\mathbb{S}^1 \times \mathbb{D}^2$

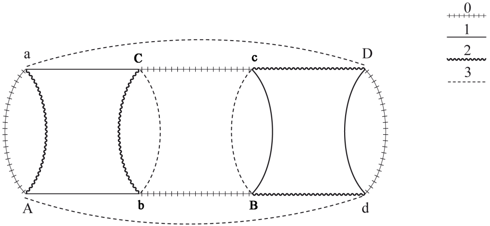
	colour 0	**colour 1**	**colour 2**	**colour 3**
vertex a	D	C	A	0
vertex A	d	b	a	0
vertex b	C	A	C	B
vertex B	c	c	d	b
vertex c	B	B	D	0
vertex C	b	a	b	0
vertex d	A	D	B	0
vertex D	a	d	c	0

The incidence matrix A_Γ is not the "most economical" way to identify (Γ, γ) (for example, $A(i,c) = j \Leftrightarrow A(j,c) = i$).

Moreover:

- if Γ is bipartite (i.e. it represents an orientable n-manifold), information about only one bipartition class is sufficient, for each colour $c \in \Delta_n$;
- if Γ is non-bipartite (i.e. it represents a non-orientable n-manifold), for each colour $c \in \Delta_{n-1}$ information about only one bipartition class is sufficient, while adjacencies by colour n have to be completely described.

Finally, by suitably labelling the vertices of the Γ, adjacencies by colour 0 may always be understood.



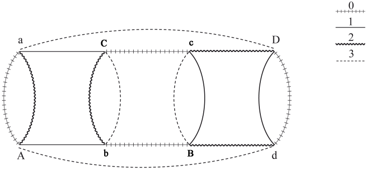
The standard crystallization of $\mathbb{S}^2 \times \mathbb{S}^1$

	colour 0	colour 1	colour 2	colour 3
vertex a		C	A	D
vertex A				
vertex b		A	C	C
vertex B				
vertex c		B	D	B
vertex C				
vertex d		D	B	A
vertex D				

The notion of *CODE*:

- for each vertex $r \in V(\Gamma)$ and for every permutation ε of $\Delta_n = \{0, 1, \ldots, n\}$, algorithmically and canonically label $V(\Gamma)$ (so that the associated incidence matrix $\bar{A}_\Gamma^{(r,\varepsilon)}$ contains "essential" elements in well-defined positions);
- if $c_{r,\varepsilon}$ is the numerical string containing in orderly way the essential elements of $\bar{A}_\Gamma^{(r,\varepsilon)}$, the *code* $code(\Gamma)$ is the lexicographic maximum among all strings $c_{r,\varepsilon}$:

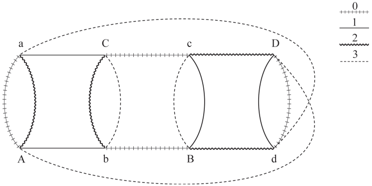
$$code(\Gamma) = max \left\{ c_{r,\varepsilon} \ / \ \begin{array}{l} r \in V(\Gamma) \\ \varepsilon \text{ permutation of } \Delta_n \end{array} \right\}$$



The standard crystallization of $\mathbb{S}^2 \times \mathbb{S}^1$

	$\varepsilon_0 = 0$	$\varepsilon_1 = 1$	$\varepsilon_2 = 3$	$\varepsilon_3 = 2$
vertex a		C	D	A
vertex A				
vertex b		A	C	C
vertex B				
vertex c		B	B	D
vertex C				
vertex d		D	A	B
vertex D				

$$code(\Gamma) \ = \ CABD \ DCBA \ ACDB$$



The standard crystallization of $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$

	$\varepsilon_0 = 0$	$\varepsilon_1 = 1$	$\varepsilon_2 = 3$	$\varepsilon_3 = 2$
vertex a		C	D	A
vertex A				a
vertex b		A	C	C
vertex B				D′
vertex c		B	B	d′
vertex C				b
vertex d′=D		D	A	c
vertex D′=d				B

$$code(\Gamma) \;=\; CABD\;\; DCBA\;\; ACdc\;\; aDbB$$

The notion of code allows to detect *colour-isomorphic* graphs, i.e. graphs isomorphic up to permutation of the vertex set AND up to permutation of the colour set:

Theorem [C.-Gagliardi 2001]

(Γ, γ) and (Γ', γ') are colour-isomorphic if and only if

$$code(\Gamma) = code(\Gamma').$$

The notion of code is very useful in order to produce automatic catalogues of PL n-manifolds via crystallizations.

In the CLOSED case, it is necessary:

- to proceed inductively on dimension n;
- to perform sphere-recognition at every step
  (*an* $(n+1)$*-coloured graph* (Γ,γ) *represents an n-manifold if and only if* $\Gamma_{\hat{c}}$ *represents the* $(n-1)$*-sphere* $\mathbb{S}^{n-1}$, $\forall c \in \Delta_n$).

Advantages in dimension $n = 3$:

- (Γ,γ) is a crystallization of a 3-manifold M^3 iff:
  - i) $\Gamma_{\hat{c}}$ is connected, $\forall c \in \Delta_3$;
  - ii) $g_{01} + g_{02} + g_{03} = 2 + p$;
  - iii) $\forall \epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$, $g_{\epsilon_0\epsilon_1} = g_{\epsilon_2\epsilon_3}$.
- all closed connected 3-manifolds may be represented by *rigid crystallizations*.

The generating algorithm was implemented in C++ programs starting from $\mathcal{S}^{(2p)}$ with $1 \leq p \leq 15$; the output data are presented in the following table according to the number of vertices.

2p	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$\#\mathcal{C}^{(2p)}$	1	0	0	1	0	1	1	3	4	23	44	262	1252	7760	56912
$\#\tilde{\mathcal{C}}^{(2p)}$	0	0	0	0	0	0	1	1	1	9	12	88	480	2790	21804

Table: rigid crystallizations up to 30 vertices

CLASSIFICATION ALGORITHM

After the generation process, suitable moves on gems, translating the PL-homeomorphism of the represented manifolds, are applied to develop a classification procedure which allows to detect crystallizations of the same manifold:

- *dipole moves*;
- *generalized dipole moves* (defined only for $n = 3$);
- *switching of ρ-pairs* (preserving the homeomorphism class, up to connected sum with handles).

The “admissible moves” are used to subdivide a given list of rigid crystallizations into equivalence classes, so that:

$$cl(\Gamma) = cl(\Gamma') \implies |K(\Gamma)| = |K(\Gamma')|$$

Remark: *Note that we have no theoretical proof that*

$$|K(\Gamma)| = |K(\Gamma')| \implies cl(\Gamma) = cl(\Gamma').$$

Nevertheless, by experimental results, the above implication is true for all elements of the catalogues $\mathcal{C}^{(2p)}$, $\tilde{\mathcal{C}}^{(2p)}$ *(with* $1 \leq p \leq 15$*), with respect to a suitably chosen set* $\bar{\mathcal{S}}$ *of admissible moves.*

In dimension 3 the above automatic partition into equivalence classes succeeds to distinguish topologically all manifolds represented by the generated catalogues:[2]

Proposition [C.-Cristofori 2008]

There exists a one-to-one correspondence between the set of classes of $\mathbf{C}^{(30)}$ (resp. $\tilde{\mathbf{C}}^{(30)}$) produced by the classification program and the set of orientable (resp. non-orientable) 3-manifolds admitting a coloured triangulation with at most 30 tetrahedra.

[2] For each positive integer p, we denote by $\mathbf{C}^{(2p)}$ (resp. $\tilde{\mathbf{C}}^{(2p)}$) the catalogue of all rigid bipartite (resp. non bipartite) crystallizations of order $\leq 2p$ arising from the generating algorithm.

Proposition [C.-Cristofori 2008]

There are exactly 110 closed prime orientable 3-manifolds, having a coloured triangulation with at most 30 tetrahedra.
They are:

- fifty-five elliptic 3-manifolds;
- thirty-nine non-elliptic Seifert 3-manifolds (in particular, two torus bundles with Nil geometry);
- four torus bundles with Sol geometry;
- two manifolds of type $(K^2 \tilde{\times} I) \cup (K^2 \tilde{\times} I)/A \quad (A \in GL(2;\mathbb{Z})$, $\det(A) = -1)$, with Sol geometry;
- seven non-geometric graph manifolds;
- three hyperbolic Dehn-fillings (of the complement of link 6^3_1).

The details both about the implementation of the classification algorithm (performed by the C++ program Γ-*class*) and about the obtained results, are available at the WEB page

http://cdm.unimo.it/home/matematica/casali.mariarita/CATALOGUES.htm

The distribution of prime manifolds in $\mathbf{C}^{(30)}$ with respect to *Matveev complexity* and *geometry*:

complexity	1	2	3	4	5	6	7	8	9	10
lens	**2/2**	**3/3**	**6/6**	**10/10**	0/20	0/36	0/72	0/136	0/272	0/528
other elliptic	-	**1/1**	**1/1**	**4/4**	**11/11**	14/25	0/45	0/78	0/142	0/270
$\mathbb{E}^3$	-	-	-	-	-	**6/6**	-	-	-	-
Nil	-	-	-	-	-	**7/7**	3/10	2/14	0/15	0/15
$\mathbb{H}^2 \times \mathbb{R}$	-	-	-	-	-	-	-	0/2	-	0/8
$\widetilde{SL}_2(\mathbb{R})$	-	-	-	-	-	-	13/39	5/162	2/513	0/1416
Sol	-	-	-	-	-	-	4/5	2/9	0/23	0/39
non-geometric	-	-	-	-	-	-	**4/4**	1/35	2/185	0/777
hyperbolic	-	-	-	-	-	-	-	-	2/4	1/25
TOTAL	**2/2**	**4/4**	**7/7**	**14/14**	11/31	27/74	24/175	10/436	6/1154	1/3078

Proposition [C. 1999] - [Bandieri-Cristofori-Gagliardi 2009]

There are exactly sixteen closed prime non-orientable 3-manifolds, having a coloured triangulation with at most 30 tetrahedra. They are:

- the 3-manifolds $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^2$ and $\mathbb{RP}^2 \times \mathbb{S}^1$, with $\mathbb{S}^2 \times \mathbb{R}$ geometry;
- the four non-orientable euclidean 3-manifolds;
- the torus bundles $TB\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $TB\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $TB\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ and $TB\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$, with Sol geometry;
- the Seifert fibered spaces $(\mathbb{RP}^2; (2,1), (3,1))$, $(\overline{D}; (2,1), (3,1))$, $(T^2/o_2; (2,1))$, $(K^2; (2,1))$ and $(K^2/n_3; (2,1))$, with $\mathbb{H}^2 \times \mathbb{R}$ geometry;
- the non-geometric graph manifold $(\mathbb{A}; (2,1)) \cup (\mathbb{A}; (2,1))/\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The distribution of prime manifolds in $\tilde{\mathbf{C}}^{(30)}$ with respect to Matveev complexity and geometry:

complexity	0	1	2	3	4	5	6	7	8	9
$\mathbb{S}^2 \times \mathbb{R}$	**1/1**	**1/1**	-	-	-	-	-	-	-	-
$\mathbb{E}^3$	-	-	-	-	-	-	**4/4**	-	-	-
$\mathbb{H}^2 \times \mathbb{R}$	-	-	-	-	-	-	-	**2/2**	0/8	3/25
Sol	-	-	-	-	-	-	**1/1**	**1/1**	**2/2**	0/2
non-geometric	-	-	-	-	-	-	-	-	-	1/6
TOTAL	**1/1**	**1/1**	**0**	**0**	**0**	**0**	**5/5**	**3/3**	2/10	4/33

Two ideas for improving the generation process:

- Simplify the considered catalogues, by means of additional conditions on the representing objects (for example by considering only **rigid cluster-less crystallizations**, which represent ALL 3-manifolds).
- Improve the implementation by making use of a refined **parallel version** of the generating algorithm.

WORK IN PROGRESS: Toward 4-dimensional catalogues

Preliminary problem
In dimension 4, the generation of crystallization catalogues implies the *previous generation of all gems (not necessarily crystallizations!) representing 3-dimensional spheres up to a fixed order*.

For the first segment of the catalogue, the recognition of gems representing $\mathbb{S}^3$ is easily faced and solved by dipole eliminations, since:

no rigid crystallization of $\mathbb{S}^3$ exists
(different from the "trivial" one, with order two)
with less than 24 vertices.

WORK IN PROGRESS: Toward 4-dimensional catalogues

The generation choice

In order to avoid the explosion of data, in dimension 4 a great attention must be paid to the choice of the representing set.

Proposition [Bandieri-Gagliardi 2011]

Each closed connected PL 4-manifold M admits a *rigid* crystallization. Moreover, if M is handle-free, it admits a rigid crystallization of minimal order.

As a consequence:

- the catalogue of crystallizations representing closed 4-manifolds may be restricted to **rigid crystallizations lacking in 2-dipoles**;
- the catalogue of *(non-contracted) gems representing* $\mathbb{S}^3$ *and lacking in* ρ_3*-pairs* may be considered as input data of the 4-dimensional generating process.

WORK IN PROGRESS: Toward 4-dimensional catalogues

The first output of the generating program:

2p	2	4	6	8	10	12	14	16	18	20
$\Gamma_{\hat{4}}(\mathbb{S}^3)$	1	0	0	9	39	400	5255	95870	1994952	45654630
$\Gamma(M^4)$ **lacking in 2-dipoles**	1	0	0	1	0	0	1109	4512	44803	47623129

Table: number of gems of $\mathbb{S}^3$ lacking in ρ_3-pairs
and number of rigid crystallizations of 4-manifolds lacking in 2-dipoles,
according to vertex number

WORK IN PROGRESS: Toward 4-dimensional catalogues

Remark: A 4-dimensional crystallization catalogue - where classification is performed up to PL-homeomorphism - might provide examples of different PL 4-manifolds belonging to the same TOP-homeomorphism class.

The *gem-complexity* of a closed n-manifold M^n is the non-negative integer $k(M^n) = p - 1$, $2p$ being the minimum order of a crystallization of M^n.

Crystallization theory easily implies:

$$k(M^4) \geq 3\beta_2(M^4)$$

for each simply connected closed 4-manifold M^4.

WORK IN PROGRESS: Toward 4-dimensional catalogues

Relation $k(M^4) \geq 3\beta_2(M^4)$, combined with the up-to-date topological classification of simply connected PL 4-manifolds allows to prove:

Proposition (to appear)

If M^4 is a simply connected closed PL 4-manifold with $k(M^4) \leq 65$, then M^4 is TOP-homeomorphic to either

$$(\#_r \mathbb{CP}^2)\#(\#_{r'} - \mathbb{CP}^2) \quad with \quad r + r' = \beta_2(M^4)$$

or

$$\#_s(\mathbb{S}^2 \times \mathbb{S}^2) \quad with \quad s = \frac{1}{2}\beta_2(M^4)$$

WORK IN PROGRESS: Toward 4-dimensional catalogues

In fact, the important results by

- [Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982), 357-453]
- [Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. 18 (1983), 279-315]
- [Furuta, *Monopole equation and the* $\frac{11}{8}$ *conjecture*, Math. Res. Lett. 8 (2001), 279-291]

ensure that only intersection forms of type

$$r[1] \oplus r'[-1] \qquad \text{or} \qquad s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are allowed if $k(M^4) \leq 65$, since forms of type

$$\pm 2nE_8 \oplus s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are proved to represent a PL 4-manifold only if $s > 2n$ (and hence, only PL 4-manifolds with $\beta_2 \geq 22$ occur in this case).

WORK IN PROGRESS: Toward 4-dimensional catalogues

RESULTS BY THE FIRST SEGMENT OF 4-DIMENSIONAL CATALOGUES:

Proposition (to appear)

- $\mathbb{S}^4$ is the only closed connected (PL) 4-manifold with gem-complexity 0.
- No closed connected (PL) 4-manifold M^4 exists with $1 \leq k(M^4) \leq 2$.
- The complex projective plane $\mathbb{CP}^2$ is the only closed connected (PL) 4-manifold with $k(M^4) = 3$.
- $k(\mathbb{S}^1 \times \mathbb{S}^3) = k(\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3) = 4$; moreover, no closed connected handle-free (PL) 4-manifold M^4 exists with $k(M^4) = 4$.
- No closed connected (PL) 4-manifold M^4 exists with $k(M^4) = 5$.

WORK IN PROGRESS: Toward 4-dimensional catalogues

RESULTS BY 4-DIMENSIONAL CATALOGUES (UP TO 18 VERTICES): THE NON-ORIENTABLE CASE

Proposition (to appear)

- No closed connected handle-free non-orientable (PL) 4-manifold M^4 exists with $k(M^4) \leq 6$.
- The real projective space $\mathbb{RP}^4$ is the only closed connected prime non-orientable (PL) 4-manifold M^4 with $k(M^4) = 7$.
- No closed connected handle-free non-orientable (PL) 4-manifold M^4 exists with $8 \leq k(M^4) \leq 9$.

WORK IN PROGRESS: Toward 4-dimensional catalogues

RESULTS BY THE 4-DIMENSIONAL CATALOGUES (FROM 14 TO 18 VERTICES): THE ORIENTABLE CASE

Proposition (to appear)

- $k(\mathbb{S}^2 \times \mathbb{S}^2) = 6$; moreover, $k(\mathbb{CP}^2 \# \mathbb{CP}^2) = k(\mathbb{CP}^2 \# (-\mathbb{CP}^2)) = 6$, too.
- If M^4 is a closed connected handle-free orientable (PL) 4-manifold with $6 \leq k(M^4) \leq 9$,) then M^4 is simply-connected and TOP-homeomorphic to either $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \mathbb{CP}^2$ or $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$.

3. Complexity estimations

By making use of the strong connection existing in dimension 3 between gems and Heegaard diagrams, a 3-manifold invariant based on crystallization theory - called *GM-complexity* - has been introduced and proved to be an upper bound for the Matveev complexity of each compact 3-manifold.

- M.R. C., *Computing Matveev's complexity of non-orientable 3-manifolds via crystallization theory*, Topology Appl. **144** (2004), 201-209.
- M.R. C., *Estimating Matveev's complexity via crystallization theory*, Discrete Math. **307** (2007), 704-714.
- M.R. C. - P. Cristofori, *Computing Matveev's complexity via crystallization theory: the orientable case*, Acta Appl. Math. **92** (2006), 113-123.
- M.R. C. - P. Cristofori - M. Mulazzani, *Complexity computation for compact 3-manifolds via crystallizations and Heegaard diagrams*, Topology and its Applications (2012), to appear.
- M.R. C. - P. Cristofori, *Computing Matveev's complexity via crystallization theory: the boundary case*, preprint 2012.

3. Estimating Matveev complexity via Heegaard diagrams

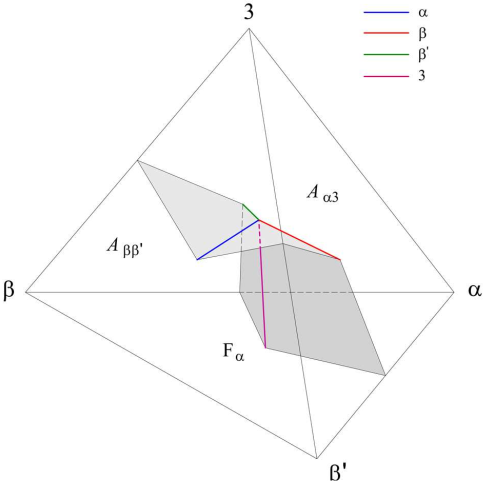
If $\mathcal{H} = (S, v, w)$ is a Heegaard diagram of M, then a special spine of M exists, whose true vertices are the intersection points of the curves of the two systems v and w, with the exception of those lying on the boundary of a region of $S - \{v \cup w\}$.

Hence:

$$c(M) \leq n - m,$$

where $n =$ number of intersection points between v and w
and $m =$ number of intersection points contained in $\bar{R}$.

$\Delta_3 = \{\alpha, \beta, \beta', 3\}$



- $K_{\alpha 3}$ (resp. $K_{\beta\beta'}$) = 1-dimensional subcomplex of $K(\Gamma)$ generated by the $\{\alpha, 3\}-$ (resp. $\{\beta, \beta'\}-$) labelled vertices.
- H_α = the largest 2-dimensional subcomplex of $K'(\Gamma)$ not intersecting the barycentric subdivisions of $K_{\alpha 3}$ e $K_{\beta\beta'}$.
- $F_\alpha = |H_\alpha|$.

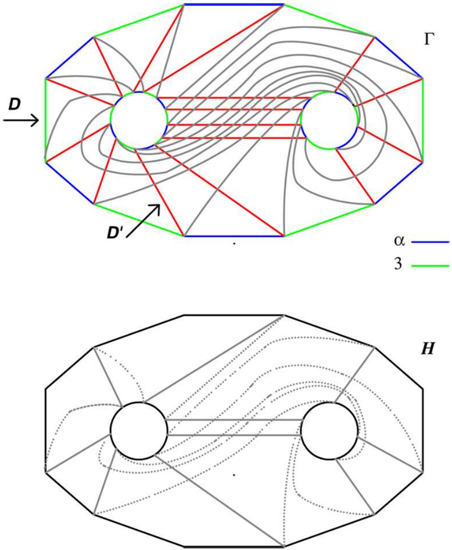
The surface F_α splits $K(\Gamma)$ into two polyhedra $\mathcal{A}_{\alpha,3}$ and $\mathcal{A}_{\alpha',\beta'}$, whose intersection is exactly F_α :

- $\mathcal{A}_{\alpha 3} \searrow K_{\alpha 3}$ (resp. $A_{\beta\beta'} \searrow K_{\beta\beta'}$) $\Longrightarrow$ is an handlebody.
- $\mathcal{A}_{\alpha 3} \cap \mathcal{A}_{\beta\beta'} = \partial\mathcal{A}_{\alpha 3} \cap \partial\mathcal{A}_{\beta\beta'} = F_\alpha$.
- If Γ is a crystallization, let D be a $\{\beta, \beta'\}$-coloured cycle and D' a $\{\alpha, 3\}$-coloured cycle.

$$\Downarrow$$

$(F_\alpha, \Gamma_{\beta\beta'} \setminus D, \Gamma_{\alpha 3} \setminus D')$ is a Heegaard diagram of M.

Example: Poincarè homology sphere



GM-complexity

GM-complexity of a crystallization

$$c_{GM}(\Gamma) = min\{\#V(\Gamma) - \#(V(D) \cup V(D') \cup V(\Xi)) \ / \ \alpha \in \Delta_3, \\ D \in \Gamma_{\beta\beta'}, \ D' \in \Gamma_{\alpha 3}, \ \Xi \in \mathcal{R}_\alpha(D, D')\}$$

GM-complexity of a closed 3-manifold

$$c_{GM}(M) = min\{c_{GM}(\Gamma) \mid \Gamma \text{ crystallization of } M\}$$

Remark:
The computation of $c_{GM}(\Gamma)$ is only based on the combinatorial structure of the edge-coloured graph Γ; hence, it may be determined in an algorithmic way (*GM-COMPLEXITY program*).

For details, see:

http://cdm.unimo.it/home/matematica/casali.mariarita/about_cgm.htm

By making use of this program, an estimation of $c(M)$ has been obtained for all manifolds involved in existing crystallization catalogues.

Conjecture

For every closed connected 3-manifold M,

$$c(M) = c_{GM}(M).$$

GM-complexity: non-contracted case

GM-complexity of a gem

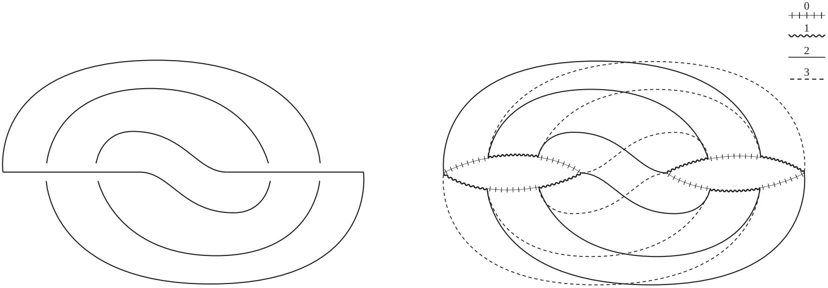
$$c_{GM}(\Gamma) = min\{\#V(\Gamma) - \#(V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi)) \ /$$
$$\alpha \in \Delta_2,$$
$$\mathcal{D} \text{ set of } \{\beta, \beta'\} - \text{cycles dual to a maximal tree of } K_{\alpha 3},$$
$$\mathcal{D}' \text{ set of } \{\alpha, 3\} - \text{cycles dual to a maximal tree of } K_{\beta\beta'},$$
$$\Xi \in \mathcal{R}_\alpha(\mathcal{D}, \mathcal{D}')\}$$

Complexity estimations [C. 2007]

The notion of *GM*-complexity, combined with the widely investigated relationships between crystallization theory and other representation methods for 3-manifolds, has allowed to obtain direct estimations of the Matveev complexity for several classes of manifolds, significantly improving former results.

Proposition. Let $M_2(L)$ be the (unique) 2-fold covering of $\mathbb{S}^3$ branched on link L and let $\overline{L}$ be a p-bridge projection of L with k crossing points ($k \geq p \geq 2$). If $\overline{L}$ admits a bridge $\bar{\beta}$ with order m_b and an independent arc $\bar{\alpha}$ with relative order m_a (i.e. an arc not consecutive to $\bar{\beta}$ and containing $m_a \geq 0$ crossing points not belonging to $\bar{\beta}$), then

$$c(M_2(L)) \leq 2(k + p - m_b - m_a - 3) \leq 4k - 8.$$

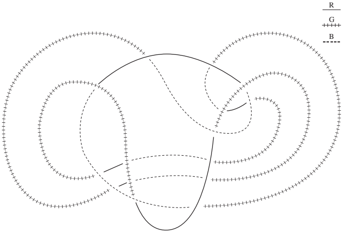


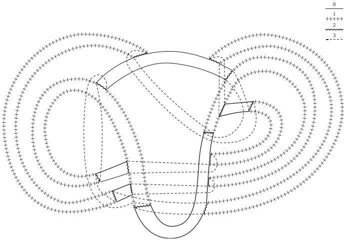
[M. Ferri, *Crystallisations of 2-fold branched coverings of* $\mathbb{S}^3$, Proc. Amer. Math. Soc. **73** (1979), 271-276.]

Complexity estimations [C. 2007]

Proposition. Let $M^3(K,\omega)$ be the 3-fold simple covering of $\mathbb{S}^3$ branched on knot K with monodromy ω and let $(\bar{K},\omega)$ be a 3-coloured diagram of knot K associated to the pair (K,ω), with k crossing points and an order m arc. Then

$$c(M^3(K,\omega)) \ \leq \ 2k - 2(m+2).$$





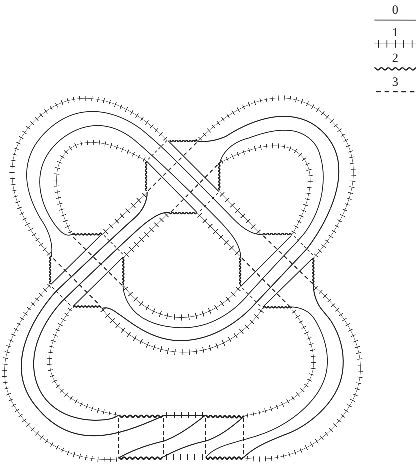
[M.R. C., *Coloured knots and coloured graphs representing 3-fold simple coverings of* S^3, Discrete Math. **137** (1995), 87-98.]

Complexity estimations [C. 2007]

Proposition. Let $M^3(L,d)$ be the 3-manifold obtained by Dehn surgery on the framed link (L,d) and let $\bar{L}$ be a planar (connected) diagram of L with $l \geq 1$ components and k crossing points. If $\bar{L}$ admits a region of order m whose boundary involves components $L_{j_1}, \dots, L_{j_s}$ $(1 \leq s \leq l)$ of L, then

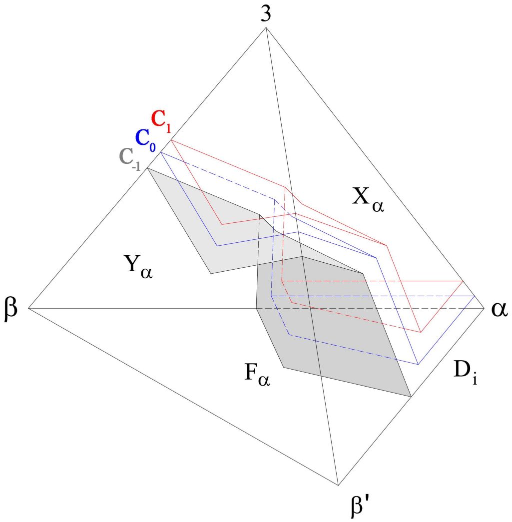
$$c(M^3(L,d)) \leq 6k + 2t - 4l - 2(m-1) - \sum_{p=1,\dots,s} t_{j_p},$$

where $t = \sum_{i=1,\dots,l} t_i$ and $t_i = |d_i - w(\bar{L}_i)|, \ \forall i = 1, \dots, l.$



[M.R. C., *From framed links to crystallizations of bounded 4-manifolds*, J. Knot Theory Ramifications **9** (2000), 443-458.]

case $\partial M \neq \emptyset$



- $\partial \mathcal{A}_{\alpha 3} \cap \partial \mathcal{A}_{\beta\beta'} = F_\alpha$
- $\partial \mathcal{A}_{\alpha 3} \cap \partial M = \bigcup_{i=1}^{r} \mathbb{D}_i$
- $\partial \mathbb{D}_i = lk(v_i, (\partial K)')$
  v_i $\alpha-$labelled vertex of ∂K
  $\bigcup_{i=1}^{r} \partial \mathbb{D}_i = \partial F_\alpha$
- $S_\alpha = F_\alpha \cup (\bigcup_{i=1}^{r} \mathbb{D}_i)$
- $C = S_\alpha \times [-1, +1]$ collar of S_α in $\mathcal{A}_{\alpha 3}$
  $C^- = S_\alpha \times [-1, 0]$
  $C^+ = S_\alpha \times [0, +1]$
  $C_i = S_\alpha \times \{i\}$

$$X_\alpha = \overline{\mathcal{A}_{\alpha 3} \setminus C^-}$$

$$Y_\alpha = \mathcal{A}_{\beta\beta'} \cup C^-$$

- X_α is an handlebody obtained from C^+ by attaching 2-handles to C_1 along the $\{\beta, \beta'\}$-coloured cycles of Γ (dual to the edges of $K_{\alpha 3}$).
- Y_α is a compression body[3], obtained from C^- by attaching 2-handles to C_{-1} along the $\{\alpha, 3\}$-coloured cycles of Γ (dual to edges of $K_{\beta\beta'}$ not belonging to ∂K).

$$\Downarrow$$

$(S_\alpha, X_\alpha, Y_\alpha)$ is a *(generalized) Heegaard splitting* of M.

[3] A *compression body* is a 3-manifold with boundary obtained from a $F \times I$ (F surface) by attaching 2-handles and 3-handles to $F \times \{1\}$.

GM-complexity of a gem with boundary

(a) $\mathcal{D}$ = set of $\{\beta, \beta'\}$- coloured cycles dual to a maximal tree of $K_{\alpha 3}$.

(b) $G_{\beta\beta'}$ = graph obtained from $K_{\beta\beta'}$ by contracting to one point p_i (for each $i = 1, \dots, r$) the vertices of $K_{\beta\beta'}$ belonging to the i-th component of ∂K.

(c) $\mathcal{D}'$ = set of $\{\alpha, 3\}$-coloured cycles dual to the edges of a subgraph $\bar{G}$ of $G_{\beta\beta'}$ such that $\bar{G}$ is a union of trees containing all vertices of $G_{\beta\beta'}$ and, $\forall i, j, \ i \neq j$, the vertices p_i and p_j belong to different connected components of $\bar{G}$.

GM-complexity of a gem with boundary

$$c_{GM}(\Gamma) = min\{\#V(\Gamma) - \#(V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi)) \ / \\ \alpha \in \Delta_2, \\ \mathcal{D}, \mathcal{D}' \ \text{ satisfying (a) and (c)} \\ \Xi \in \mathcal{R}_\alpha(\mathcal{D}, \mathcal{D}')\}$$

GM-complexity of a 3-manifold with boundary

GM-complexity of a 3-manifold with boundary

$$c_{GM}(M) = \min\{c_{GM}(\Gamma) \mid \Gamma \ \text{ gem of } M\}$$

By definition itself:

Proposition [C.-Cristofori 2012]

For each compact 3-manifold M,

$$c(M) \leq c_{GM}(M)$$

WORK IN PROGRESS: estimations via c_{GM} (boundary case)

By making use of the notion of *GM-complexity* for 3-manifolds with boundary, we hope to obtain - via graph theoretical construction of the associated gems - improvements for existing estimations of Matveev complexity for some interesting classes of bounded 3-manifolds.

The first efforts will take into considerations

knot (or link) complements.