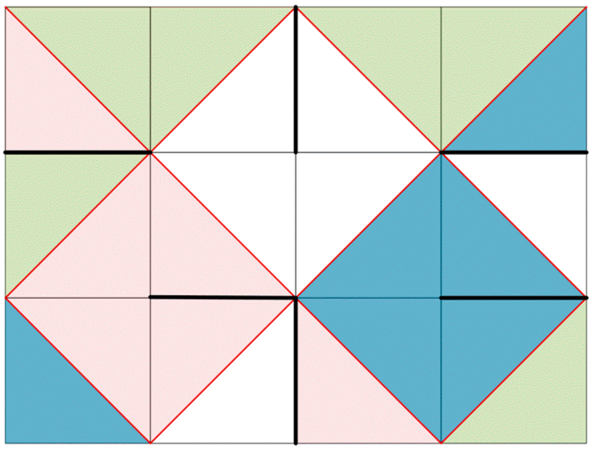
A MATHEMATICAL TRIBUTE

to Professor

José María Montesinos Amilibia



DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
FACULTAD DE CIENCIAS MATEMÁTICAS – UCM

A MATHEMATICAL TRIBUTE
to Professor José María Montesinos Amilibia

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Professor José María Montesinos Amilibia

Oh my knots!

This volume contains the contributions presented by several colleagues as a tribute to the mathematical and human qualities of José María Montesinos Amilibia on the occasion of his seventieth birthday. The editors would like to express their thanks to the contributors and their very especial gratitude to José María for his example through many years of scientific and personal contact.

Marco Castrillón
Elena Martín-Peinador
José M. Rodríguez-Sanjurjo
Jesús M. Ruiz

Muchas gracias, José María

Este es un acto importante para los que estamos aquí, algunos de los cuales conocen a José María Montesinos desde hace más de 50 años. Hemos venido para darle las gracias por muchas cosas, pero sobre todo por su ejemplo, el ejemplo de cómo es posible hacer cosas valiosas en tiempo difíciles o muy difíciles, como fueron los que vivió en la primera etapa de su investigación, desde 1967 hasta que se fue a Estados Unidos en 1976. Aquí, en la facultad, solo y por sus propios medios, realizó los que, según su propio testimonio, son sus mejores trabajos. Esto se puede constatar también en su página web de la Academia donde cita los que cree son sus trabajos más interesantes, todos pertenecientes a aquella época. Algunos publicados en revistas muy relevantes, como el Bulletin of the AMS, otros en revistas más modestas, pero todos conteniendo resultados excelentes. También la crítica, en particular el Mathematical Reviews, ha sido explícita en su reconocimiento de la calidad de los trabajos de aquel periodo. Posteriormente Montesinos conoció y aprendió mucho de Thurston y de otros: Casson, Kirby, Matsumoto, Edwards, Siebenmann y Fico González Acuña, y siempre ha manifestado su admiración y reconocimiento hacia ellos. Evidentemente, su actividad científica posterior se ha beneficiado enormemente de estos contactos y de colaboraciones como la que ha mantenido con María Teresa Lozano y Mike Hilden a lo largo de tantos años.

En sentido inverso se puede decir que otros aprendieron no menos de él. Esto se ha podido comprobar en el flujo continuo de visitantes de todo el mundo que han venido al departamento para aprender. Éste es el sentido opuesto al que nos ha llevado a la mayoría a ir fuera para adquirir conocimiento.

Montesinos ha obtenido un gran reconocimiento científico. Él ha valorado y agradecido el reconocimiento, científico y humano, de sus colegas. Especialmente importante para él fue el que le dio Ralph Fox en su etapa inicial. Gracias al apoyo que le prestó, pudo saber que iba en la buena dirección y que sus resultados eran significativos. Desgraciadamente, Fox falleció antes del primer viaje de José María a Estados Unidos y no pudo conocerle personalmente. Sin embargo Montesinos no ha buscado los focos, no se siente cómodo cuando es objeto de la atención pública (espero que este acto sea una excepción). En el año 1992 se celebró en París el primer congreso de la Sociedad Matemática Europea. Era una ocasión importante, el lanzamiento de esta sociedad, y fueron invitados diez conferenciantes plenarios de la talla de Arnold, Donaldson y Mumford. Uno de ellos era Montesinos, quizá algunos recuerden los carteles que anunciaban el congreso, en los que él aparecía en esa lista. Sin embargo surgió un problema de financiación y José María renunció a la invitación. Aparte de las razones económicas, creo percibir que esa decisión estuvo motivada por una humildad básica que siempre me ha parecido ver en su personalidad.

Montesinos disfruta del contacto humano, y los demás perciben cuánto pueden ganar estando a su lado y hablando con él. En cualquier reunión científica se puede advertir su popularidad y la relación amistosa de que disfruta con mucha gente. Pero, al mismo tiempo, necesita de la soledad, el retiro. Para su vida interior y para sus teoremas. Los teoremas que fueron hechos en su juventud y los que sigue haciendo en las montañas de Guadarrama y de Gredos. Recientemente nos decía en un seminario que uno de sus últimos resultados le había costado cinco salidas al monte. Le deseo muchas más salidas al monte, muchos más buenos teoremas y que siga viniendo por aquí para contárnoslos. Muchas gracias de nuevo, José María.

Madrid, 8 de septiembre de 2015 *José Manuel Rodríguez-Sanjurjo*
Facultad de Ciencias Matemáticas, UCM

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Contributions on the occasion of JMMA's 70 anniversary

Classifying PL 4-manifolds via crystallizations: results and open problems

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To our friend José María, with admiration and gratitude.

Abstract

Crystallization theory is a graph-theoretical representation method for compact PL-manifolds of arbitrary dimension, which makes use of a particular class of edge-coloured graphs, which are dual to coloured (pseudo-)triangulations. The purely combinatorial nature of crystallizations makes them particularly suitable for automatic generation and classification, as well as for the introduction and study of graph-defined invariants for PL-manifolds.

The present survey paper focuses on the 4-dimensional case, presenting up-to-date results about the PL classification of closed 4-manifolds, by means of two such PL invariants: *regular genus* and *gem-complexity*.

Open problems are also presented, mainly concerning different classification of 4-manifolds in TOP and DIFF=PL categories, and a possible approach to the 4-dimensional Smooth Poincaré Conjecture.

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Key words: PL 4-manifold, coloured graph, coloured triangulation, regular genus, gem-complexity, simple crystallization, semi-simple crystallization.

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1. Introduction

Crystallization theory is a representation theory for PL-manifolds by means of edge-coloured graphs, which are dual 1-skeletons of particular (pseudo)-triangulations. These graphs are called *crystallizations* if the associated triangulations have the minimum possible number of vertices.

The peculiarity of this method consists in its universality: in fact, it enables to represent and study the whole class of PL-manifolds, without restrictions about dimension, boundary property and orientability (despite what happens with other classical representation theories, whose extension beyond dimension three appears to be quite difficult to achieve, or restricted to particular hypotheses).

The possibility of performing a purely combinatorial approach to general PL-manifolds is of particular interest in dimension greater or equal to four, where the difference between the categories TOP and PL (and DIFF, if $n \geq 5$) must be taken into account. For example, it is very useful to have combinatorial moves on the representing objects, which realize the PL-homeomorphism (and not only the TOP-homeomorphism) between the represented manifolds, or to define PL invariants of the manifolds (possibly distinguishing different PL structures on the same TOP-manifold), whose computation can be performed directly on the combinatorial objects.

Within crystallization theory, both tools are available: suitable sets of moves on edge-coloured graphs exist, which enable to recognize different crystallizations of the same PL n-manifold, and some interesting graph-defined invariants for PL-manifolds have been introduced and deeply studied.

In particular:

- the *gem-complexity* $k(M^n)$ of a PL n-manifold M^n is the integer $p-1$, where $2p$ is the minimum order of a crystallization of M^n;

- the *regular genus* $\mathcal{G}(M^n)$ of an orientable (resp. non-orientable) PL n-manifold M^n is defined as the minimum genus (resp. half the minimum genus) of a surface into which a crystallization of M^n regularly embeds (see Section 2).

Note that the regular genus extends to higher dimension the classical concept of Heegaard genus of a 3-manifold, and succeeds in characterizing PL-spheres and disks of arbitrary dimension (see Subsection 4.1); on the other hand, gem-complexity is the natural invariant involved in possible generation and analysis of catalogues of PL n-manifolds via crystallizations.[1]

The present survey paper focuses on the 4-dimensional closed case, and presents in a unified view updated results - most of them very recent and still in publication -

[1] In dimension three, the automatic generation and analysis of catalogues of 3-dimensional crystallizations for increasing values of their vertex-number has already produced the classification of all closed 3-manifolds up to gem-complexity 14: see [4], [18].

about topological and PL classifications of closed PL 4-manifolds according to both gem-complexity and regular genus. Some new results are also included.

As regards regular genus, the classifying results related to closed PL 4-manifolds mainly concern the case of "low" regular genus (Proposition 4.2) and the case of "restricted gap" between the regular genus and the rank of the fundamental group of the manifold (Proposition 4.3). All of them are based essentially on the existence of particular types of handle decompositions induced by crystallizations of the involved PL 4-manifolds, and make use of an important result by Montesinos [45] ensuring the uniqueness of the boundary identification of two handlebodies (see Subsection 4.1).

On the other hand, the classifying results via gem-complexity are based on the development of an effective algorithm for the automatic generation and classification of the catalogue of 4-manifold crystallizations up to a fixed number of vertices (see Subsection 4.2). Theorem 4.6 summarizes the complete PL classification of closed orientable (resp. non-orientable) PL 4-manifolds up to gem-complexity 8 (resp. up to gem-complexity 9), due to [20], and contains also a new statement, concerning the partial - and in progress - classification of crystallizations with 20 vertices.

The difficulty of the exact calculation of both the regular genus and gem-complexity for any given PL 4-manifold, makes the search for significant lower and upper bounds a relevant task. In Section 3 a result - recently obtained in [6] - is presented, yielding sharp lower bounds for both invariants, by means of the Euler characteristic and the rank of the fundamental group of the involved 4-manifold (Theorem 3.1). These bounds turn out to be very useful to improve estimation for regular genus and gem-complexity of product 4-manifolds (see Subsection 3.2), and to obtain a new proof of the TOP classification of simply-connected 4-manifolds up to regular genus 43 and gem-complexity 65 (Theorem 3.5).

Section 5 is devoted to the so called *semi-simple crystallizations*, introduced in [6] so that the represented PL 4-manifolds attain the above lower bounds. The additivity of both gem-complexity and regular genus with respect to connected sum is proved for such a class of PL 4-manifolds, which comprehends all "standard" ones and their connected sums.

Note that additivity of regular genus for closed PL 4-manifolds was conjectured in [31] and has been proved to imply - by a theorem of Wall - the 4-dimensional Smooth Poincaré Conjecture. Therefore, the identification of classes of manifolds for which the property holds is an interesting open problem (see Section 6.2).

Other further developments, mainly concerning different classification of 4-manifolds in TOP and DIFF=PL categories, are reviewed in the last section of the paper. In particular, it is discussed the possible application of the classification algorithm to the crystallizations arising from the two known 16-vertices and 17-vertices triangulations of the $K3$-surface obtained in [26] and [46] respectively.

2. Basic notions of crystallization theory

In the present work, when not otherwise explicitly specified, we will consider only closed, connected piecewise linear manifolds of dimension $n = 4$ (simply referred to as "PL 4-manifolds"). Therefore, although edge-coloured graphs are a representation tool for the whole class of PL-manifolds, in this section we will briefly review basic notions and results of the theory with respect to this particular case.

A *5-coloured graph (without boundary)* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a regular multigraph (i.e. it may include multiple edges, but no loop) of degree five and $\gamma : E(\Gamma) \to \Delta_4 = \{0, 1, 2, 3, 4\}$ is a proper edge-coloration (i.e. it is injective when restricted to the set of edges incident to any vertex of Γ).

The elements of the set Δ_4 are called the *colours* of Γ; thus, for every $i \in \Delta_4$, an *i-coloured edge* is an element $e \in E(\Gamma)$ such that $\gamma(e) = i$. For every $i, j, k \in \Delta_4$ let $\Gamma_{\hat{\imath}}$ (resp. Γ_{ijk}) (resp. Γ_{ij}) be the subgraph obtained from (Γ, γ) by deleting all the edges of colour i (resp. $c \in \Delta_4 - \{i, j, k\}$) (resp. $c \in \Delta_4 - \{i, j\}$). The connected components of $\Gamma_{\hat{\imath}}$ (resp. Γ_{ijk}) (resp. Γ_{ij}) are called *$\hat{\imath}$-residues* (resp. *$\{i, j, k\}$-coloured residues*) (resp. *$\{i, j\}$-coloured cycles*) of Γ, and their number is denoted by $g_{\hat{\imath}}$ (resp. g_{ijk}) (resp. g_{ij}).

A 5-coloured graph (Γ, γ) is called *contracted* iff, for each $i \in \Delta_4$, the subgraph $\Gamma_{\hat{\imath}}$ is connected (i.e. iff $g_{\hat{\imath}} = 1 \ \forall i \in \Delta_4$).

Every 5-coloured graph (Γ, γ) may be thought of as the combinatorial visualization of a 4-dimensional labelled pseudocomplex $K(\Gamma)$, which is constructed in the following way:

- for each vertex $v \in V(\Gamma)$, take a 4-simplex $\sigma(v)$, with vertices labelled $0, 1, 2, 3, 4$;
- for each j-coloured edge between v and w ($v, w \in V(\Gamma)$), identify the 3-dimensional faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertex labelled j, so that equally labelled vertices coincide.

In case $K(\Gamma)$ triangulates a PL 4-manifold M, then (Γ, γ) is called a *gem* (gem = graph encoded manifold) *representing* M.

In the following, for sake of conciseness, we will write Γ instead of (Γ, γ), when there is no ambiguity with regard to the edge-coloration.

The following proposition summarizes some useful results which come directly from the above construction.

Proposition 2.1 *If Γ is an order $2p$ gem of a PL 4-manifold M, then:*

(a) M is orientable iff Γ is bipartite;

(b) there is a bijection between i-labelled vertices (resp. 1-simplices whose vertices are labelled $\Delta_4 - \{i, j, k\}$) (resp. 2-simplices whose vertices are labelled $\Delta_4 - \{i, j\}$) of $K(\Gamma)$ and $\hat{\imath}$-residues (resp. $\{i, j, k\}$-coloured residues) (resp. $\{i, j\}$-coloured cycles) of Γ;

(c) $\chi(|K(\Gamma)|) = -3p + \sum_{i,j} g_{ij} - \sum_{i,j,k} g_{ijk} + \sum_i g_i$;

(d) $2g_{ijk} = g_{ij} + g_{ik} + g_{jk} - p$ *for each triple* $(i,j,k) \in \Delta_4$;

(e) for each distinct $i,j,k \in \Delta_4$, *there exists a presentation of* $\pi_1(M)$ *whose generators are in bijection with the connected components of* Γ_{ijk} *but one.*

A gem representing a PL 4-manifold M is a *crystallization* of M if it is also a contracted graph; by the above property (b), this is equivalent to require that the associated pseudocomplex $K(\Gamma)$ contains exactly five vertices (one for each label $i \in \Delta_4$). Pezzana Theorem and its subsequent improvements prove that every PL-manifold admits a crystallization (see [32]).

The following proposition allows to characterize crystallizations of PL 4-manifolds among 5-coloured graphs.

Proposition 2.2 *A 5-coloured graph* Γ *is a crystallization of a PL* 4*-manifold if and only if, for every* $c \in \Delta_4$, $\Gamma_{\hat{c}}$ *is connected and represents* $\mathbb{S}^3$.

Catalogues of crystallizations of PL manifolds have been obtained both in dimension three (see [41], [17] and [18] for the 3-dimensional orientable case and [14], [16] and [4] for the non-orientable one) and four [20]. As mentioned in Section 1, they are constructed with respect to a suitable graph-defined PL invariant, which measures how "complicated" the representing combinatorial object is[2].

Definition 1 Given a PL n-manifold M^n, its *gem-complexity* is the non-negative integer $k(M^n) = p - 1$, where $2p$ is the minimum order of a crystallization of M^n.

An *h-dipole* ($1 \leq h \leq 4$) of a 5-coloured graph Γ is a subgraph of order two of Γ, having h edges coloured by $\{c_1, \ldots, c_h\}$, such that its vertices belong to different connected components of $\Gamma_{\Delta_4 - \{c_1,\ldots,c_h\}}$.

A *ρ-pair* in Γ is a pair of equally coloured edges both belonging to at least three common bicoloured cycles of Γ.

It is proved in [20, Proposition 20] that, if M is a *handle-free* PL 4-manifold (i.e.: if it admits neither the orientable nor the non-orientable $\mathbb{S}^3$-bundle over $\mathbb{S}^1$ as a connected summand), then $k(M) = p - 1$, where $2p$ is the order of a crystallization of M with no dipoles and no ρ-pairs.

Crystallizations with these properties are called *rigid dipole-free crystallizations*; they are exactly the elements considered in the existing crystallization catalogues in dimension four.[3]

Another graph-based invariant for PL n-manifolds, called *regular genus*, is related to some of the most interesting results of crystallization theory[4]. It was introduced

[2] In dimension three, the relations between this invariant and the well-known Matveev's complexity have been widely investigated: see [16], [17], [19] and [22].

[3] A slightly modified definition of *rigidity* is required in 3-dimensional crystallization catalogues.

[4] See, for example, [11], [12] and [25] for 4-dimensional results, [23] and [13] for 5-dimensional ones.

in [35] and its definition relies on the existence of a particular type of embedding into a surface for gems of arbitrary dimension.

As far as the 4-dimensional case is concerned, it is well-known that, if Γ is an order $2p$ crystallization of an orientable (resp. non-orientable) PL 4-manifold M, then for every cyclic permutation $\varepsilon = (\varepsilon_o, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = 4)$ of Δ_4 there exists a so-called *regular embedding*[5] $i_\varepsilon : |\Gamma| \to F_\varepsilon$, where F_ε is the closed orientable (resp. non-orientable) surface of genus $\rho_\varepsilon(\Gamma)$ (resp. $2\rho_\varepsilon(\Gamma)$), where $\rho_\varepsilon(\Gamma)$ may be directly computed by the following formula (see [35] for details and suitable extensions in general dimension).

$$\sum_{i\in\mathbb{Z}_5} g_{\varepsilon_i\varepsilon_{i+1}} - 3p = 2 - 2\rho_\varepsilon(\Gamma). \tag{2.1}$$

Definition 2 The *regular genus* of a bipartite (resp. non-bipartite) 5-coloured graph Γ is defined as the minimum genus (resp. half the minimum genus) of a surface into which Γ regularly embeds:

$$\rho(\Gamma) = \min_{\varepsilon}\{\rho_\varepsilon(\Gamma)\};$$

the *regular genus* of a PL 4-manifold M is defined as the minimum regular genus of a crystallization of M:

$$\mathcal{G}(M) = \min\{\rho(\Gamma) \;/\; (\Gamma,\gamma) \text{ crystallization of } M\}.$$

Finally, we recall that, given two 5-coloured graphs Γ_1 and Γ_2 representing PL 4-manifolds M_1 and M_2 respectively, for any choice of $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$, it is possible to construct a new 5-coloured graph $\Gamma_1\#_{v_1,v_2}\Gamma_2$, called a *graph connected sum* of Γ_1 and Γ_2, by deleting v_1 and v_2 and welding the hanging edges according to their colours.

$\Gamma_1\#_{v_1,v_2}\Gamma_2$ turns out to be a gem of one of the two possible connected sums of M_1 and M_2 (see [32] for details).

3. Lower bounds and their consequences

3.1. Lower bounds for regular genus and gem-complexity

The following result - recently obtained by Basak and Casali - is very useful to investigate PL-manifolds of dimension four by means of the two invariants regular genus and gem-complexity.

Theorem 3.1 [6] *Let M be a PL 4-manifold with $rk(\pi_1(M)) = m$. Then:*

$$k(M) \;\geq\; 3\chi(M) + 10m - 6;$$

$$\mathcal{G}(M) \;\geq\; 2\chi(M) + 5m - 4.$$

[5] By short, it is a cellular embedding whose regions are bounded by the images of $\{\varepsilon_i, \varepsilon_{i+1}\}$-coloured cycles, for each $i \in \mathbb{Z}_5$.

Roughly speaking, we can say that the proof of the inequality concerning gem-complexity is based on the Dehn-Sommerville equations in dimension four, applied to the contracted pseudo-triangulation $K(\Gamma)$ of M associated to any crystallization of M, by making use of the relations $g_{ijk} \geq m+1$ for any distinct $i,j,k \in \Delta_4$ (coming from the assumption $rk(\pi_1(M)) = m$ and Proposition 2.1(e)).

On the contrary, the inequality concerning regular genus makes use of the following crucial steps (see [6, Theorem 1] for details):

- By the first inequality of Theorem 3.1, the minimum possible order of a crystallization of M is $2\bar{p} = 6\chi(M) + 10(2m-1)$; hence, any crystallization (Γ, γ) of M has $\#V(\Gamma) = 2\bar{p} + 2q$ for some non-negative integer q.
- For any distinct $i,j,k \in \Delta_4$, the assumption $rk(\pi_1(M)) = m$ implies $g_{ijk} = (m+1) + t_{ijk}$, where $t_{ijk} \in \mathbb{Z}$, $t_{ijk} \geq 0$ and $\sum_{0 \leq i<j<k \leq 4} t_{ijk} = q$.
- By Proposition 2.1(d), the ten relations $g_{ij} + g_{ik} + g_{jk} = 2g_{ijk} + (\bar{p}+q)$ $(0 \leq i < j < k \leq 4)$ give rise to a linear system of equations (in the numbers of the different bicoloured cycles) which may be solved, so to obtain the following lower bound (surprisingly not depending from q) for the regular genus of Γ with respect to any cyclic permutation ε of Δ_4: $\rho_\varepsilon(\Gamma) \geq \frac{2(\bar{p}-1)-5m}{3}$.
- Since both the crystallization Γ of M and the permutation ε of Δ_4 are arbitrary, the second inequality of Theorem 3.1 easily follows.

3.2. Regular genus and gem-complexity of product 4-manifolds

In [6], Theorem 3.1 is applied in order to significantly improve some lower bounds for the regular genus of PL 4-manifolds, which have been proved by various authors via different techniques; meanwhile, similar lower bounds are obtained also for gem-complexity.

Proposition 3.2 [6] *For any closed 3-manifold M^3 such that $\pi_1(M^3)$ is a finite abelian group, then:*

$$\mathcal{G}(M^3 \times \mathbb{S}^1) \geq 5rk(\pi_1(M^3)) + 1 \qquad \text{and} \qquad k(M^3 \times \mathbb{S}^1) \geq 10rk(\pi_1(M^3)) - 6.$$

In particular:

$$\mathcal{G}(L(p,q) \times \mathbb{S}^1) \geq 6 \qquad \text{and} \qquad k(L(p,q) \times \mathbb{S}^1) \geq 4.$$

Proposition 3.3 [6] *Let T_g (resp. U_h) denote the orientable (resp. non-orientable) surface of genus $g \geq 0$ (resp. $h \geq 1$). Then:*

$$\mathcal{G}(T_g \times T_r) \geq 8gr + 2g + 2r + 4 \qquad \text{and} \qquad k(T_g \times T_r) \geq 12gr + 8g + 8r + 6;$$

$$\mathcal{G}(T_g \times U_h) \geq 4gh + 2g + h + 4 \qquad \text{and} \qquad k(T_g \times U_h) \geq 6gh + 8g + 4h + 6;$$

$$\mathcal{G}(U_h \times U_k) \geq 2hk + h + k + 4 \qquad \textit{and} \qquad k(U_h \times U_k) \geq 3hk + 4h + 4k + 6.$$

In particular:

$$\mathcal{G}(\mathbb{S}^2 \times T_g) \geq 2g + 4 \qquad \textit{and} \qquad k(\mathbb{S}^2 \times T_g) \geq 8g + 6;$$

$$\mathcal{G}(\mathbb{S}^2 \times U_h) \geq h + 4 \qquad \textit{and} \qquad k(\mathbb{S}^2 \times U_h) \geq 4h + 6.$$

Moreover, the last inequality of Proposition 3.3 concerning regular genus (resp. gem-complexity), together with the existence of a genus five (resp. order 24) crystallization of $\mathbb{S}^2 \times \mathbb{RP}^2$ depicted in [6, Figure 3], allows the exact calculation of the regular genus (resp. an estimation with "strict range" of the gem-complexity) of the involved 4-manifold:

Proposition 3.4 [6]

$$\mathcal{G}(\mathbb{S}^2 \times \mathbb{RP}^2) = 5 \qquad \textit{and} \qquad k(\mathbb{S}^2 \times \mathbb{RP}^2) \in \{10, 11\}.$$

3.3. TOP classification of simply-connected 4-manifolds via regular genus and gem-complexity

A direct application of Theorem 3.1 to the case of simply-connected PL 4-manifolds, combined with well-known results on TOP simply-connected 4-manifolds, yields the following interesting result related to the topological classification of simply-connected PL 4-manifolds with respect both to *gem-complexity* and to *regular genus*:[6]

Theorem 3.5 [20] *Let M be a simply-connected PL 4-manifold. If either $k(M) \leq 65$ or $\mathcal{G}(M) \leq 43$, then M is TOP-homeomorphic to*

$$(\#_r \mathbb{CP}^2)\#(\#_{r'}(-\mathbb{CP}^2)) \quad \textit{or} \quad \#_s(\mathbb{S}^2 \times \mathbb{S}^2),$$

where $r + r' = \beta_2(M)$, $s = \frac{1}{2}\beta_2(M)$ and $\beta_2(M)$ is the second Betti number of M.

Proof. Since M is assumed to be simply-connected, Theorem 3.1 yields:

$$k(M) \geq 3\beta_2(M) \tag{3.1}$$

and

$$\mathcal{G}(M) \geq 2\beta_2(M). \tag{3.2}$$

[6]Note that the proof presented in the present survey paper is easier than the original one (contained in [20, Proposition 20 and Proposition 23]), since it directly makes use of the inequalities derived from Theorem 3.1.

Now, the classical theorems of Freedman and Donaldson (see [33]) about the TOP classification of simply-connected closed 4-manifolds, together with more recent results by Furuta [34], ensure that intersection forms of type

$$\pm 2nE_8 \oplus s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

do represent a PL 4-manifold only if $s > 2n$; hence, only PL 4-manifolds with $\beta_2(M) \geq 22$ occur in this case. The thesis directly follows from the fact that both $k(M) \leq 65$ and $\mathcal{G}(M) \leq 43$ imply $\beta_2(M) \leq 21$; so, only intersection forms of the two simplest types are allowed:

$$r[1] \oplus r'[-1] \qquad or \qquad s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $r + r' = \beta_2(M)$ or $s = \frac{1}{2}\beta_2(M)$. □

4. PL classification via regular genus and gem-complexity

4.1. Classifying results via regular genus

The first, important property of regular genus consists in the possibility of recognizing spheres (and disks[7]) of arbitrary dimension, in full analogy with well-known low-dimensional characterizations.

Theorem 4.1 [31] *For every closed PL n-manifold M^n, with $n \geq 2$,*

$$\mathcal{G}(M^n) = 0 \quad \Longleftrightarrow \quad M^n \cong \mathbb{S}^n$$

Actually, the regular genus shares many properties with other low-dimensional genera (see [3] for a survey on results in general dimension, including the boundary case); for example, for every PL n-manifold M^n, $n \geq 3$, $\mathcal{G}(M^n)$ is a non-negative integer invariant, so that $\mathcal{G}(M^n) \geq rk(\pi_1(M^n))$.

Starting from the above results, many efforts have been spent in order to investigate the relation existing between the "PL structure" of a manifold M^n and its regular genus $\mathcal{G}(M^n)$, in order to yield classifying results via regular genus in PL category and dimension n, both in the closed and in the boundary case. The best results have been achieved in dimension 4 and 5, and concern the case of "low" regular genus, the case of "restricted gap" between the regular genus of the manifold and the regular genus of its boundary, and the case of "restricted gap" between the regular genus and the rank of the fundamental group of the manifold.

[7]Several results of this subsection admit suitable extensions to PL manifolds with non-empty boundary. However, for sake of conciseness, we restrict the statement of Theorem 4.1 to the closed case, as everywhere in the paper.

The following Propositions 4.2 and 4.3 exactly deal with the first and third cases in the closed 4-dimensional setting, while the second one, which concerns PL 4-manifolds with boundary, is out of the scope of the present survey.

From now on, $\mathbb{S}^1 \times \mathbb{S}^3$ (resp. $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3$) (resp. $\mathbb{S}^1 \otimes \mathbb{S}^3$) will denote the orientable (resp. non-orientable) (resp. either orientable or non-orientable) $\mathbb{S}^3$-bundle over $\mathbb{S}^1$.

Proposition 4.2 [27, 28, 29]

a) Let M be a closed orientable 4-manifold; then:

$$\mathcal{G}(M) = \rho \leq 3 \quad \Longrightarrow \quad M \cong \begin{cases} \#_\rho(\mathbb{S}^1 \times \mathbb{S}^3) \\ \#_{\rho-2}(\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{CP}^2 \end{cases}$$

b) Let M be a closed non-orientable 4-manifold; then:

$$\mathcal{G}(M) = \rho \leq 2 \quad \Longrightarrow \quad M \cong \#_\rho(\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3)$$

Proposition 4.3 [12, 25]

a) Let M be a closed (orientable or non-orientable) 4-manifold; then:

$$\mathcal{G}(M) = rk(\pi_1(M)) = \rho \quad \Longleftrightarrow \quad M \cong \#_\rho(\mathbb{S}^1 \otimes \mathbb{S}^3)$$

b) Let M be a closed 4-manifold; then:

- $\mathcal{G}(M) \neq rk(\pi_1(M)) \implies \mathcal{G}(M) - rk(\pi_1(M)) \geq 2$
- *$\mathcal{G}(M) - rk(\pi_1(M)) = 2$ and $\pi_1(M) = *_m\mathbb{Z} \iff M \cong \#_m(\mathbb{S}^1 \otimes \mathbb{S}^3)\#\mathbb{CP}^2$*
- *No closed 4-manifold M exists with $\mathcal{G}(M) - rk(\pi_1(M)) = 3$ and $\pi_1(M) = *_m\mathbb{Z}$.*

In short, the proofs of Propositions 4.2 and 4.3 are based on the fact that, for every crystallization Γ of a PL 4-manifold M, the associated triangulation $K(\Gamma)$ gives rise to a suitable handle-decomposition of M, which reflects the combinatorial properties of Γ.
First of all, we recall that every closed PL 4-manifold M admits a handle-decomposition

$$M = H^{(0)} \cup (H_1^{(1)} \cup \cdots \cup H_{r_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{r_2}^{(2)}) \cup (H_1^{(3)} \cup \cdots \cup H_{r_3}^{(3)}) \cup H^{(4)}$$

where $H^{(0)} = \mathbb{D}^4$ and each p-handle $H_i^{(p)} = \mathbb{D}^p \times \mathbb{D}^{4-p}$ $(1 \leq p \leq 4,\ 1 \leq i \leq r_p)$ is endowed with an an embedding (called *attaching map*) $f_i^{(p)} : \partial\mathbb{D}^p \times \mathbb{D}^{4-p} \to \partial(H^{(0)} \cup \ldots (H_1^{(p-1)} \cup \cdots \cup H_{r_{p-1}}^{(p-1)}))$.

In particular, for any crystallization Γ of a PL 4-manifold M and for any partition $\{\{i,j,k\},\{r,s\}\}$ of Δ_4, then M admits a decomposition of type $M = N(i,j,k) \cup_\phi$

$N(r,s)$, where $N(i,j,k)$ (resp. $N(r,s)$) denotes a regular neighbourhood of the subcomplex $K(i,j,k)$ (resp. $K(r,s)$) of $K(\Gamma)$ generated by the vertices labelled $\{i,j,k\}$ (resp. $\{r,s\}$) and ϕ is a boundary identification.

The hypotheses assumed about regular genus in most of the cases of the above statements imply the associated handle-decomposition to lack in 2-handles, since the subcomplex $K(i,j,k)$ collapses to a graph; this fact allows to recognize the 4-manifold M as a connected sum of copies of $\mathbb{S}^1 \otimes \mathbb{S}^3$, by means of an important result by Montesinos [45] ensuring the uniqueness of the boundary identification of two handlebodies (see the quoted papers for details).

Under different assumptions, the associated handle-decomposition contains exactly one 2-handle and no 3-handle (i.e.: $K(i,j,k)$ consists of two triangles with common boundary, possibly with some more "free edges", in bijection with the "free edges" constituting $K(r,s)$); so, the attachment of the unique 2-handle has to give rise to a spherical boundary, and hence a $\mathbb{CP}^2$ component is surely obtained, via a well-known result by Gordon-Luecke about surgery on knots.

Remark 1 Note that the classification of orientable (resp. non-orientable) 4-manifolds with regular genus four (risp. three) is given in [29, Theorem 2] (resp. [29, Theorem 4]) only up to TOP-homeomorphism[8], while the classification of non-orientable 4-manifolds with regular genus four is not known.

However, Proposition 4.3 yields the following partial results, concerning the PL classification of a PL 4-manifold M with $rk(\pi_1(M)) = m$:

- if $\mathcal{G}(M) = 3$ and $m = 3$, then M is PL-homeomorphic to $\#_3(\mathbb{S}^1 \otimes \mathbb{S}^3)$;
- if $\mathcal{G}(M) = 4$, $m = 2$ and the fundamental group of M is free, then M is PL-homeomorphic to $\mathbb{CP}^2 \#_2(\mathbb{S}^1 \otimes \mathbb{S}^3)$.

Moreover, no PL 4-manifold exists with $\mathcal{G}(M) = 3$ (resp. $\mathcal{G}(M) = 4$) and $m \in \{0, 2\}$ (resp. $m = 3$).

In Subsection 5.3 we will "almost complete"[9] the PL classification up to regular genus four, within the class of PL 4-manifolds admitting semi-simple crystallizations.

4.2. Catalogues of crystallizations

By Proposition 2.2, the generation of catalogues of crystallizations of PL 4-manifolds with a fixed number of vertices $2p$ requires the prior generation and recognition of all gems (with $2p$ vertices) representing the 3-sphere. In fact, any order $2p$ crystallization of a PL 4-manifold can be obtained from such a gem by adding the 4-coloured edges.

[8] Recall that, in virtue of Theorem 3.5, the classification of simply-connected 4-manifolds in category TOP is now trivial, up to regular genus 43.

[9] Only the case $\mathcal{G}(M) = 4$ and $rk(\pi_1(M)) = 2$, with not free fundamental group, remains open.

However, this kind of procedure is very intensive even for a low number of vertices and requires large computing resources. A way to face this problem is to find combinatorial configurations in the graphs, which can be eliminated without changing the manifold. Examples of such configurations are dipoles and ρ-pairs.

As pointed out in Section 2 the restriction of the catalogues to rigid dipole-free crystallizations does not affect their completeness.

Furthermore, an efficient catalogue must not contain crystallizations, whose associated triangulations coincide. This problem has been solved by associating to each crystallization Γ its *code*, i.e. a numerical string which identifies Γ up to *colour-isomorphisms* (i.e. isomorphisms of graphs which preserve colours up to a permutation of Δ_4: see [41], [24]).

Catalogues of (rigid dipole-free) crystallizations of PL 4-manifolds have been generated up to 20 vertices and the represented manifolds have been completely classified up to 18 vertices.

In order to describe the algorithms for generation and classification of these catalogues, we need first to fix some notations.

For each $p \geq 1$, let $\mathcal{C}^{(2p)}$ (resp. $\tilde{\mathcal{C}}^{(2p)}$) denote the catalogue of all not colour-isomorphic rigid dipole-free bipartite (resp. non-bipartite) crystallizations of 4-manifolds with $2p$ vertices.

Note that if $\Gamma \in \mathcal{C}^{(2p)} \cup \tilde{\mathcal{C}}^{(2p)}$, then $\Gamma_{\hat{4}}$ is a (not necessarily contracted, nor rigid) 4-coloured graph representing $\mathbb{S}^3$ and lacking in ρ-pairs involving three bicoloured cycles. Let $S^{(2p)}$ denotes the set of such 4-coloured graphs[10]: the generation of $\mathcal{C}^{(2p)}$ and $\tilde{\mathcal{C}}^{(2p)}$ is performed by adding 4-coloured edges to the elements of $S^{(2p)}$, in all possible ways so as to obtain crystallizations of 4-manifolds. Duplicates are then eliminated by comparing their codes.

Actually, not all attachments must be tried: the condition of representing a manifold imposes combinatorial restrictions on the set of possible "uncompleted" graphs (i.e. graphs obtained from an element of $S^{(2p)}$ by attaching less than p 4-coloured edges).

These conditions allow to implement a branch and bound technique to prune the tree of possible attachments (see [44] for details) and reduce considerably both the computation time and the size of the resulting catalogues[11].

Table 1 shows information about the catalogues $\mathcal{C}^{(2p)}$ and $\tilde{\mathcal{C}}^{(2p)}$ ($p \leq 10$), which have been obtained in [20] by the above algorithm.

[10] $S^{(2p)}$ is constructed by a suitable adaptation of the 3-dimensional generation algorithm; recognition of the 3-sphere is performed by comparison with the 3-dimensional catalogue.

[11] The generation algorithm has been implemented through a parallelization strategy ([44]); it ran on CINECA's high-performance clusters due to the opportunities granted by an Italian Supercomputing Resource Allocation (ISCRA) project.

2p	2	4	6	8	10	12	14	16	18	20
$\#S^{(2p)}$	1	0	2	9	39	400	5.255	95.870	1.994.962	45.654.630
$\#\ \mathcal{C}^{(2p)}$	1	0	0	1	0	0	1.109	4.511	44.803	47.623.129
$\#\ \tilde{\mathcal{C}}^{(2p)}$	0	0	0	0	0	0	0	1	0	0

Table 1

Note that the unique rigid dipole-free crystallization of $\mathcal{C}^{(2)}$ (resp. of $\mathcal{C}^{(8)}$) is the standard crystallization of $\mathbb{S}^4$ (resp. $\mathbb{CP}^2$: see [36]), while the unique non-bipartite rigid dipole-free crystallization appearing up to 20 vertices is the standard one of $\mathbb{RP}^4$ with 16 vertices ([37]).

In order to face the problem of classification of the PL-manifolds appearing in catalogues of crystallizations, in [20] it is described a heuristic procedure based on combinatorial moves on graphs which do not change ("up to handles") the represented manifold and preserve the properties of rigidity and absence of dipoles.

We point out that all concepts and results involved in the procedure hold in each dimension $n \geq 3$; therefore the whole classifying algorithm is introduced in [20] for general PL n-manifolds. Nevertheless, for sake of simplicity, in the present survey paper we will describe it only in the 4-dimensional setting.

Let us call *admissible* any sequence of combinatorial moves which transforms a rigid dipole-free crystallization of a PL 4-manifold M, into a rigid dipole-free crystallization of a PL 4-manifold M' such that $M \cong_{PL} M' \#_h (\mathbb{S}^1 \otimes \mathbb{S}^3)$ $(h \geq 0)$.

Moreover, for any rigid dipole-free crystallization Γ of M and any admissible sequence ϵ, let $\theta_\epsilon(\Gamma)$ denote the (rigid dipole-free) crystallization of M' obtained by applying the admissible sequence ϵ to Γ.

Given a list X of rigid dipole-free crystallizations and a set $\mathcal{S}$ of admissible sequences, it is then possible to subdivide X into equivalence classes with regard to $\mathcal{S}$ by defining the class of $\Gamma \in X$ with respect to $\mathcal{S}$ as:

$$cl_{\mathcal{S}}(\Gamma) = \{\Gamma' \in X \ / \ \exists \epsilon, \epsilon' \in \mathcal{S},\ \theta_\epsilon(\Gamma) \text{ and } \theta_{\epsilon'}(\Gamma') \text{ have the same code}\}$$

Therefore, given $\Gamma, \Gamma' \in X$, if $cl_{\mathcal{S}}(\Gamma) = cl_{\mathcal{S}}(\Gamma')$, then there exist $h, k \in \mathbb{N} \cup \{0\}$ such that $|K(\Gamma)| \cong_{PL} M\#_h(\mathbb{S}^1 \otimes \mathbb{S}^3)$ and $|K(\Gamma')| \cong_{PL} M\#_k(\mathbb{S}^1 \otimes \mathbb{S}^3)$.

Obviously, no choice of $\mathcal{S}$ can ensure "a priori" that the above equivalence classes coincide, even up to handles, with the PL-equivalence classes of the represented 4-manifolds.

However, [20] shows the existence of a suitable set of admissible moves on 5-coloured graphs which are sufficient to classify all PL 4-manifolds admitting a crystallization with at most 18 vertices, in full analogy with what already proved in dimension three, where a similar set has been detected, yielding the classification of all 3-manifolds admitting a crystallization with at most 30 vertices.

It is well-known that insertion or elimination of a dipole (*dipole move*) preserves the PL structure of the represented manifold in any dimension (see [30] for details).

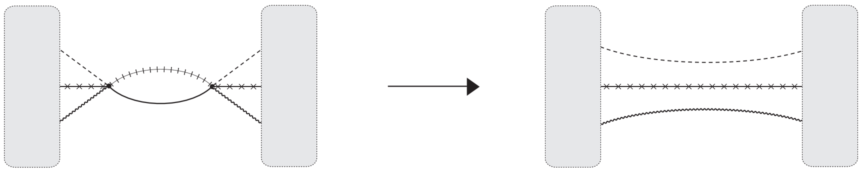

Figure 1: dipole move

Furthermore, a combinatorial move called *ρ-pair switching*, which is shown in Figure 4.2, allows to eliminate ρ-pairs.

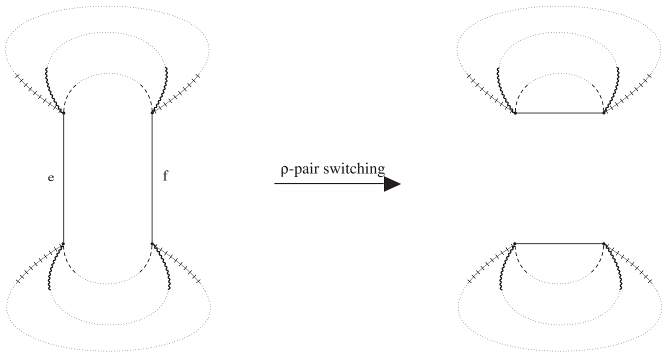

Figure 2: ρ-pair switching

The effect of ρ-pair switching on crystallizations is explained by the following result:

Proposition 4.4 [5] *Let Γ be a crystallization of a PL 4-manifold M and let Γ' be obtained by switching a ρ-pair (e, f) in Γ. Suppose that e and f share $h \geq 3$ bicoloured cycles of Γ. Then:*

(a) if $h = 3$, Γ' is a gem of M, too;

(b) if $h = 4$, Γ' is a gem of a PL 4-manifold M' such that $M \cong_{PL} M' \# (\mathbb{S}^1 \otimes \mathbb{S}^3)$.

Unfortunately, some moves which turned out to be powerful for the classification of 3-manifolds are not available in dimension greater than three. Therefore, the algorithm described in [20] makes use of other moves which were introduced by Lins and Mulazzani in [42].

Let Γ be a gem of a PL 4-manifold M.

- A *blob move* is the insertion or cancellation of a 4-dipole.
- A *t-flip* is the switching of a pair (e, f) of equally coloured edges which are both incident to an h-dipole $(1 \leq h \leq 3)$. An *s-flip* is the inverse move, i.e. the switching of a pair (e, f) of equally coloured edges where either e or f belongs to an h-dipole, which becomes an $(h-1)$-dipole after the transformation. A *flip* is either an s- or a t-flip.

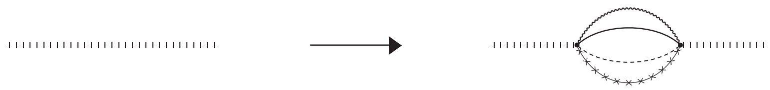

Figure 3: blob move

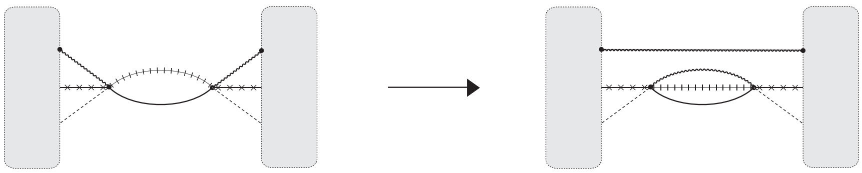

Figure 4: flip move

Flip and blob moves on a gem do not change the represented manifold as proved in [42, Proposition 3].

On the other hand, we point out that, even if two crystallizations are known to represent the same manifold, there is no algorithmic procedure to determine a sequence of blob and flip moves connecting them, nor an upper bound to the number of moves to be performed.

In order to define the set of admissible moves $\bar{\mathcal{S}}$ which have been chosen for the heuristic procedure, let us introduce some definitions and notations.

Given an order $2p$ 5-coloured graph Γ there is a natural ordering of its vertices induced by the *rooted numbering algorithm* generating its code (see [24]); so $V(\Gamma) = \{v_1, \dots, v_{2p}\}$ may be assumed.

If Γ is a rigid dipole-free crystallization of a PL 4-manifold, given $i \in \mathbb{N}_{2p} = \{1, \dots, 2p\}$, $c \in \Delta_4$, a 4-tuple $\mathbf{x} = (x_1, \dots, x_4)$ with $x_i \in \mathbb{N}_{2p}$ and a permutation τ of $\hat{c} = \Delta_4 - \{c\}$, we denote by $\theta_{i,c,\mathbf{x},\tau}(\Gamma)$ the rigid dipole-free crystallization obtained from Γ in the following way:

- insert a 4-dipole (= blob) over the c-coloured edge incident with v_i;
- for each $k \in \hat{c}$, consider, if exists, the s-flip on the pair of $\tau(k)$-coloured edges (e, f), where e belongs to the blob and f is incident to v_{x_k}; then perform the sequence of all possible s-flips of this type for increasing values of k;

- cancel dipoles and switch ρ-pairs in the resulting graph.

$\theta_{i,c,\mathbf{x},\tau}$ obviously defines an admissible sequence.

Finally, $\bar{\mathcal{S}}$ is defined as the set of all sequences $\theta_{i,c,\mathbf{x},\tau}$, where $i \in \mathbb{N}_{2p}$, $c \in \Delta_4$, $\mathbf{x}$ is a 4-tuple of elements of $\mathbb{N}_{2p}$ and τ is a permutation of $\hat{c}$.

Remark 2 As already mentioned, the above moves, as well as the classification algorithm itself, are independent from dimension. As a consequence, the set $\bar{\mathcal{S}}$ can be defined and the partition into equivalence classes with respect to $\bar{\mathcal{S}}$ can be performed on any list of crystallizations of n-manifolds, in order to prove their PL-equivalence. See [20] (or Subsection 6.1) for examples of application of the above algorithm.

4.3. Classification results in PL category

In order to obtain PL classification results, the classification algorithm, with respect to the above defined set $\bar{\mathcal{S}}$, has been implemented in a C++ program - called "Γ4-*class*" and it has been applied to the catalogues $\mathcal{C}^{(2p)}$ and $\tilde{\mathcal{C}}^{(2p)}$ with $p \leq 9$ and to the subset of $\mathcal{C}^{(20)}$ consisting of crystallizations representing manifolds with $\beta_2 \leq 2$.

The application of Γ4-*class* to the catalogue $\bigcup_{1\leq p\leq 9} \mathcal{C}^{(2p)}$ yielded the complete PL classification of the involved crystallizations as shown in the following proposition.

Proposition 4.5 [20] *There is a bijective correspondence between the set of equivalence classes of* $\bigcup_{1\leq p\leq 9}\left(\mathcal{C}^{(2p)} \cup \tilde{\mathcal{C}}^{(2p)}\right)$ *with respect to* $\bar{\mathcal{S}}$ *and the set of the represented PL 4-manifolds. Moreover, all PL 4-manifolds in the above catalogues are topologically distinct.*

By the above result and [20, Proposition 15], it has been obtained the PL classification of all orientable (resp. non-orientable) 4-manifolds with gem-complexity at most 8 (resp. at most 9), which is summarized in the following theorem. Note that the last statement appears in the present survey in a stronger form than the original result in [20]: in fact, only recently program Γ4-*class* succeeded to prove that no PL 4-manifold M with $k(M) = 9$ has $\beta_2(M) \leq 2$ (i.e. all 20 vertices crystallizations with $\beta_2(M) \leq 2$ belongs to the same class of a crystallization with few vertices).

Theorem 4.6 *Let M be a PL 4-manifold. Then:*

- $k(M) = 0 \iff$ *M is PL-homeomorphic to* $\mathbb{S}^4$;
- $k(M) = 3 \iff$ *M is PL-homeomorphic to* $\mathbb{CP}^2$;
- $k(M) = 4 \iff$ *M is PL-homeomorphic to either* $\mathbb{S}^1 \times \mathbb{S}^3$ *or* $\mathbb{S}^1 \tilde{\times} \mathbb{S}^3$;
- $k(M) = 6 \iff$ *M is PL-homeomorphic to either* $\mathbb{S}^2 \times \mathbb{S}^2$ *or* $\mathbb{CP}^2 \# \mathbb{CP}^2$ *or* $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$;

- $k(M) = 7 \iff$ *M is PL-homeomorphic to either* $\mathbb{RP}^4$ *or* $\mathbb{CP}^2\#(\mathbb{S}^1 \times \mathbb{S}^3)$ *or* $\mathbb{CP}^2\#(\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3)$;
- $k(M) = 8 \iff$ *M is PL-homeomorphic to either* $\#_2(\mathbb{S}^1 \times \mathbb{S}^3)$ *or* $\#_2(\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3)$.

Moreover:

- *no PL* 4*-manifold M exists with* $k(M) \in \{1, 2, 5\}$;
- *no exotic PL* 4*-manifold exists, with* $k(M) \leq 8$;
- *any PL* 4*-manifold M with* $k(M) = 9$ *is simply connected with second Betti number* $\beta_2(M) = 3$.

As a consequence of the (partial) analysis of the 4-dimensional crystallization catalogue $\bigcup_{1 \leq p \leq 10} \mathcal{C}^{(2p)}$, together with a suitable application of the classification program Γ4-*class*, in [20] it is also proved the existence of a rigid crystallization of $\mathbb{S}^4$, with 20 vertices and, apart from the standard order-two crystallization, it is the only rigid dipole-free crystallization of $\mathbb{S}^4$ up to 20 vertices.

5. 4-manifolds admitting simple and semi-simple crystallizations

5.1. Simple and semi-simple crystallizations

The notion of *simple crystallization* of a (simply-connected) PL 4-manifold was introduced in [7] and investigated in [21]; further, in [6], it was extended to the not simply-connected case, by introducing the concept of *semi-simple crystallization* of a PL 4-manifold.

Definition 3 A crystallization Γ of a PL 4-manifold M is called a *semi-simple crystallization of type m* if the 1-skeleton of the associated coloured triangulation contains exactly $m + 1$ 1-simplices for each pair of 0-simplices, where m is the rank of the fundamental group of M.

Semi-simple crystallizations of type 0 are called *simple crystallizations*: the 1-skeleton of their associated coloured triangulation equals the 1-skeleton of a single 4-simplex.

Remark 3 In virtue of the bijection between 1-simplices of $K(\Gamma)$ and residues of Γ with three colours (see Section 2), the above definition may be re-stated in combinatorial terms, for any crystallization Γ of a PL 4-manifold M with $rk(\pi_1(M)) = m$:

$$\Gamma \text{ is a semi-simple crystallization} \iff g_{ijk} = m + 1, \ \forall i, j, k \in \Delta_4.$$

In particular:

$$\Gamma \text{ is a simple crystallization} \iff g_{ijk} = 1, \ \forall i, j, k \in \Delta_4.$$

As a direct consequence of the proof of Theorem 3.1 (inequality concerning gem-complexity), a characterization of PL 4-manifolds admitting simple/semi-simple crystallizations easily follows, involving the relation between the invariant gem-complexity and the Euler characteristic.

Theorem 5.1 [21, 6] *A PL 4-manifold M admits semi-simple crystallizations of type m if and only if $k(M) = 3\chi(M)+10m-6$, where m is the rank of the fundamental group of M.*
In particular: A simply-connected PL 4-manifold M admits simple crystallizations if and only if $k(M) = 3\chi(M) - 6$.

In [21] (resp. in [6]) simple (resp. semi-simple) crystallizations are proved to be "minimal" both with respect to the invariant gem-complexity and with respect to the invariant regular genus. Here, we present the generalized result concerning semi-simple crystallizations, which contextually yields a lot of details about their combinatorial structure.

Theorem 5.2 [6] *Let M be a PL 4-manifold with $rk(\pi_1(M)) = m$. If M admits semi-simple crystallizations, then:*

$$k(M) = 3\chi(M) + 10m - 6;$$

$$\mathcal{G}(M) = 2\chi(M) + 5m - 4;$$

$$k(M) = \frac{3\mathcal{G}(M) + 5m}{2}.$$

Moreover, for any semi-simple crystallization Γ of M,

- $\rho_\epsilon(\Gamma) = \mathcal{G}(M) = 2\chi(M) + 5m - 4$ *for any cyclic permutation ϵ of Δ_4;*
- $\#V(\Gamma) = 2(k(M) + 1) = 6\chi(M) + 20m - 10;$
- $g_{i,j} = \chi(M) + 4m - 1$ *for any pair $i, j \in \Delta_4$;*
- $\rho_\epsilon(\Gamma_{\hat{i}}) = \frac{\mathcal{G}(M)-m}{2} = \chi(M) + 2m - 2$ *for any cyclic permutation ϵ of Δ_4 and for any color $i \in \Delta_4$.*

In the simply-connected case, the characterization of PL 4-manifolds admitting simple crystallizations via gem-complexity may be performed with respect to the second Betti number, as proved for the first time in [21, Theorem 1.1].

Theorem 5.3 [21] *Let M be a simply-connected PL 4-manifold. Then:*

$$M \text{ admits a simple crystallization iff } k(M) = 3\beta_2(M).$$

Moreover, if M admits simple crystallizations, then $\mathcal{G}(M) = 2\beta_2(M)$.

Remark 4 It is not difficult to prove that, if $\pi_1(M)$ is assumed to be trivial, then equality $\mathcal{G}(M) = 2\beta_2(M)$ implies the existence of a crystallization Γ of M and a permutation ε of Δ_4 so that $\rho_\varepsilon(\Gamma) = 2\beta_2(M)$ and $g_{\varepsilon_i\varepsilon_{i+2}\varepsilon_{i+3}} = 1\ \forall i \in \Delta_4$. However, in general, this does not imply that Γ is simple, since at least one $g_{\varepsilon_i\varepsilon_{i+1}\varepsilon_{i+2}} > 1$ may occur. For example, the analysis of the catalogue of rigid dipole-free order 16 crystallizations shows that all of them satisfy relation $\mathcal{G}(M) = 2\beta_2(M)$, while $g_{rst} = 2$ for exactly one triple $\{r,s,t\} \subset \Delta_4$ and $g_{ijk} = 1\ \forall \{i,j,k\} \neq \{r,s,t\}$.

Let us conclude the subsection by pointing out that, in the particular case of a simple crystallization, any associated handle-decomposition (see Subsection 4.1) turns out to be a so-called *special handle decomposition*, i.e. a handle-decomposition lacking in 1-handles and 3-handles (see [43, Section 3.3]). Hence, PL 4-manifolds admitting simple crystallizations may be identified by a (not dotted) framed link.[12]

Proposition 5.4 [21] *Let M be a (simply-connected) PL 4-manifold admitting simple crystallizations. Then, M admits a special handle-decomposition.*

In fact, with the same notations used in Subsection 4.1, we can notice that, if Γ is a simple crystallization, then for any partition $\{\{i,j,k\},\{r,s\}\}$ of Δ_4 the decomposition $M = N(i,j,k) \cup_\phi N(r,s)$ is of "standard" type: $K(r,s)$ consists of exactly one 1-simplex, while $K(i,j,k)$ consists of g_{rs} 2-simplices, all having the same boundary. Hence, $N(r,s) \cong_{PL} \mathbb{D}^4 = H^{(4)}$ trivially follows, while $N(i,j,k) = H^{(0)} \cup (H_1^{(2)} \cup \cdots \cup H_{g_{rs}-1}^{(2)})$ holds, where $H^{(0)} = \mathbb{D}^4$ is a "small" regular neighbourhood of one (arbitrarily fixed) 2-simplex of $K(i,j,k)$ and the 2-handles are represented by the regular neighbourhoods of the remaining 2-simplices of $K(i,j,k)$.

Remark 5 Note that the existence of a special handlebody decomposition is related to Kirby problem n. 50: *"Does every simply-connected closed 4-manifold have a handlebody decomposition without 1-handles? Without 1- and 3-handles?"*.

5.2. Computing regular genus and gem-complexity for a huge class of PL 4-manifolds

The definition of graph connected sum (see Section 2) implies that the class of PL 4-manifolds admitting simple/semi-simple crystallization is closed under connected sum.

Proposition 5.5 [7] *Let M and M' be two PL 4-manifolds admitting semi-simple crystallizations. Then, $M\#M'$ admits semi-simple crystallizations, too.*
In particular, if both M and M' admit simple crystallizations, then $M\#M'$ admits simple crystallizations, too.

[12] See [15] for relationships between crystallization theory and dotted framed link representation for PL 4-manifolds.

It is easy to check that the well-known minimal (order 10) crystallizations of $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3$, as well as the minimal (order 16) crystallization of $\mathbb{RP}^4$, are semi-simple crystallizations of type 1, while the minimal (order 2) crystallization of $\mathbb{S}^4$, the minimal (order 8) crystallization of $\mathbb{CP}^2$, the minimal (order 14) crystallization of $\mathbb{S}^2 \times \mathbb{S}^2$ are simple crystallizations. Moreover, in [7] a simple crystallization of the $K3$-surface is produced.

As a consequence, in virtue of the additivity of semi-simple crystallizations (Proposition 5.5), we have that all simply-connected PL 4-manifolds of "standard type" (see [7]) admit semi-simple crystallizations, as well as all PL 4-manifolds involved in the existing crystallization catalogues (see [20], or Section 4).

Proposition 5.6 [7] *Each PL 4-manifold with gem-complexity less than nine admits semi-simple crystallizations.*

By direct analysis of the existing 4-dimensional crystallization catalogues (see [20] or Section 4), we can compute how many simple/semi-simple crystallizations exist, for some significant PL 4-manifolds.

Proposition 5.7

- $\mathbb{S}^4$ *and* $\mathbb{CP}^2$ *admit a unique simple crystallization;*
- $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3$ *and* $\mathbb{RP}^4$ *admit a unique semi-simple crystallization (of type 1);*
- $\mathbb{S}^2 \times \mathbb{S}^2$ *admits exactly* 267 *simple crystallizations;*
- $\mathbb{CP}^2 \# \mathbb{CP}^2$ *admits exactly* 583 *simple crystallizations;*
- $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$ *admits exactly* 258 *simple crystallizations.*

From Theorem 5.2 and Proposition 5.5 it is easy to deduce the additivity of both the invariants regular genus and gem-complexity under connected sum, within the class of PL 4-manifolds admitting semi-simple crystallizations (in particular: simple crystallizations).

Theorem 5.8 [21, 7] *Let M and M' be two PL 4-manifolds admitting semi-simple crystallizations. Then:*

$$k(M \# M') = k(M) + k(M') \quad \textit{and} \quad \mathcal{G}(M \# M') = \mathcal{G}(M) + \mathcal{G}(M').$$

As a consequence, we obtain the computation of both invariants for a huge class of PL 4-manifolds.

Proposition 5.9 *Let*

$$M \cong_{PL} (\#_p \mathbb{CP}^2)\#(\#_{p'}(-\mathbb{CP}^2))\#(\#_q(\mathbb{S}^2 \times \mathbb{S}^2))\#(\#_r(\mathbb{S}^1 \otimes \mathbb{S}^3))\#(\#_s \mathbb{RP}^4)\#(\#_t K3),$$

with $p, p', q, r, s, t \geq 0$. *Then,*

$$k(M) = 3(p+p'+2q+22t)+4r+7s \quad \textit{and} \quad \mathcal{G}(M) = 2(p+p'+2q+22t)+r+3s.$$

5.3. Classification via regular genus of PL 4-manifolds admitting simple/semi-simple crystallizations

In the particular case of PL 4-manifolds admitting semi-simple crystallizations, some classification results via regular genus may be added to the general ones summarized in Subsection 4.1. The first one concerns the case of the first Betti number equal to one.

Proposition 5.10 [7] *Let M be an orientable PL 4-manifold with $\pi_1(M) = *_m\mathbb{Z}$ and $\beta_2(M) = 1$. If M admits semi-simple crystallizations, then M is PL-homeomorphic to $\#_m(\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{CP}^2$.*

Moreover, the relation between regular genus and gem-complexity for PL 4-manifolds admitting semi-simple crystallizations yields new results regarding the PL classification up to regular genus four, within the class of PL 4-manifolds admitting a semi-simple crystallization. As already pointed out in Subsection 4.1, only the case $\mathcal{G}(M) = 4$ and $rk(\pi_1(M)) = 2$, with not free fundamental group, remains open.

Proposition 5.11 *Let M be a PL 4-manifold with $rk(\pi_1(M)) = m$ which admits semi-simple crystallizations. Then:*

(a) if $\mathcal{G}(M) = 3$ and $m = 1$, then M is PL-homeomorphic to one of the following PL 4-manifolds: $\mathbb{CP}^2\#(\mathbb{S}^1 \times \mathbb{S}^3)$, $\mathbb{CP}^2\#(\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3)$, $\mathbb{RP}^4$;

(b) if $\mathcal{G}(M) = 3$ and $m = 3$, then M is PL-homeomorphic to $\#_3(\mathbb{S}^1 \otimes \mathbb{S}^3)$;

(c) if $\mathcal{G}(M) = 4$ and $m = 0$, then M is PL-homeomorphic to one of the following PL 4-manifolds: $\mathbb{CP}^2\#\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{CP}^2\#(-\mathbb{CP}^2)$;

(d) if $\mathcal{G}(M) = 4$, $m = 2$ and the fundamental group of M is free, then M is PL-homeomorphic to $\mathbb{CP}^2\#_2(\mathbb{S}^1 \otimes \mathbb{S}^3)$.

Moreover, if $\mathcal{G}(M) = 3$ (resp. $\mathcal{G}(M) = 4$), the cases $m \in \{0, 2\}$ (resp. $m \in \{1, 3\}$) cannot appear for PL 4-manifolds admitting semi-simple crystallizations.

Proof. First of all, note that, by Remark 1, only statements (a) and (c) require the hypothesis about semi-simple crystallizations, and have to be proved.

By Theorem 5.2, if M is a PL 4-manifold admitting semi-simple crystallizations, relation $k(M) = \frac{3\mathcal{G}(M)+5m}{2}$ holds. Then: in case (a) $k(M) = 7$ follows, while in case (c) $k(M) = 6$ follows. Hence, statements (a) and (c) are consequence of the PL classification of all the (orientable and non-orientable) PL 4-manifolds up to gem-complexity 8, obtained in [20] via analysis of the related crystallization catalogues (see Theorem 4.6).

Moreover, the relation $k(M) = \frac{3\mathcal{G}(M)+5m}{2}$ easily implies that the cases $\mathcal{G}(M) = 3$ (resp. $\mathcal{G}(M) = 4$) and $m \in \{0, 2\}$, (resp. $m \in \{1, 3\}$) are impossible, if M admits semi-simple crystallizations. $\Box$

General conditions excluding the existence of semi-simple crystallizations are also obtained from their combinatorial properties (Theorem 5.2).

Proposition 5.12 [7] *No PL 4-manifold M with $\mathcal{G}(M) - rk(\pi_1(M))$ odd admits semi-simple crystallizations.*
In particular: no simply-connected PL 4-manifold M with odd regular genus admits simple crystallizations.

6. Toward further developments

6.1. Different PL structures on the same TOP 4-manifold, and related problems

As it is well-known, up to now there is no classification of smooth structures on any given smoothable topological 4-manifold; on the other hand, finding non-diffeomorphic smooth structures on the same closed simply-connected topological manifold has long been an interesting problem.

Our hope is that further advances in the generation and classification of crystallization catalogues for PL 4-manifolds according to gem-complexity (see the algorithm described in [20, Section 3] and briefly summarized in Subsection 4.2) could produce examples of non-equivalent PL structures on the same TOP 4-manifold.

For example, the characterization of PL 4-manifolds admitting semi-simple crystallizations by means of gem-complexity (Theorem 5.1) has the following consequence about possible different PL structures on the same TOP 4-manifold.

Proposition 6.1 *Let M and M' be two PL 4-manifolds, with $M \cong_{TOP} M'$ and $M \ncong_{PL} M'$. If both M and M' admit semi-simple crystallization, then $k(M) = k(M')$.*

Remark 6 In particular, in the simply-connected case, the catalogue of all rigid dipole-free crystallizations of PL 4-manifolds up to a fixed gem-complexity k must contain all simple crystallizations of PL 4-manifolds whose second Betti number does not exceed $\frac{k}{3}$. Hence, the existing catalogue up to gem-complexity 9 ([20]) presents all simple crystallizations of PL 4-manifolds M with $\beta_2(M) \leq 3$.

On the other hand, in [1], infinitely many PL 4-manifolds TOP-homeomorphic but not PL-homeomorphic to $\mathbb{CP}^2\#_2(-\mathbb{CP}^2)$ are proved to exist. Hence, if at least one among them admits simple crystallizations, they have to appear in the catalogue of order 20 crystallizations, whose analysis is currently underway.

More generally, the existence of simple/semi-simple crystallizations may be related with known results and open problems about exotic structures on "standard" simply-connected PL 4-manifolds and on non-orientable PL 4-manifolds (see for example [1], [2] and [10]), by taking into account also the (obvious) finiteness property of gem-complexity. Items (a), (d), (e) of the following proposition are due to [20, 21], while items (b) and (c) are new.

Proposition 6.2

(a) Let M be $\mathbb{S}^4$ or $\mathbb{CP}^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2\#\mathbb{CP}^2$ or $\mathbb{CP}^2\#(-\mathbb{CP}^2)$; if an exotic PL structure on M exists, then the corresponding PL-manifold does not admit simple crystallizations.

(b) Let $\bar{M}$ be a PL 4-manifold TOP-homeomorphic but not PL-homeomorphic to $\mathbb{RP}^4$; then, $\bar{M}$ does not admit semi-simple crystallizations.

(c) Let M be $\mathbb{S}^1 \otimes \mathbb{S}^3$ or $\mathbb{CP}^2\#(\mathbb{S}^1 \otimes \mathbb{S}^3)$ or $\#_2(\mathbb{S}^1 \otimes \mathbb{S}^3)$; if an exotic PL structure on M exists, then the corresponding PL-manifold does not admit semi-simple crystallizations.

(d) Let $\bar{M}$ be a PL 4-manifold TOP-homeomorphic but not PL-homeomorphic to $\mathbb{CP}^2\#_2(-\mathbb{CP}^2)$; then, either $\bar{M}$ does not admit simple crystallizations, or $\bar{M}$ admits an order 20 simple crystallization.

(e) Let $r \in \{3,5,7,9,11,13\} \cup \{r = 4n-1 \ / \ n \geq 4\} \cup \{r = 4n-2 \ / \ n \geq 23\}$; then, infinitely many simply-connected PL 4-manifolds with $\beta_2 = r$ do not admit simple crystallizations.

As pointed out by [7, Corollary 8.3], the existence of pairs of simply-connected "standard" PL 4-manifolds which are TOP-homeomorphic but not PL-homeomorphic has a consequence involving simple crystallizations, too.

Proposition 6.3 [7] *A pair of simple crystallizations Γ, Γ' exists, such that*

$$|K(\Gamma)| \cong_{TOP} |K(\Gamma')| \quad but \quad |K(\Gamma)| \not\cong_{PL} |K(\Gamma')|.$$

In fact, by Kronheimer and Mrowka (see [40]), it is known that $K3\#(-\mathbb{CP}^2) \cong_{TOP} \#_3(\mathbb{CP}^2)\#_{20}(-\mathbb{CP}^2)$ but $K3\#(-\mathbb{CP}^2) \not\cong_{PL} \#_3(\mathbb{CP}^2)\#_{20}(-\mathbb{CP}^2)$. Hence, to obtain a pair of simple crystallizations proving the above statement, it is sufficient to make graph connected sums of suitable copies of the (known) simple crystallizations of $K3$ and $\mathbb{CP}^2$ (see Subsection 5.2).

Remark 7 If Γ' and Γ'' is a pair of simple crystallizations of M' and M'' respectively, with $M' \cong_{TOP} M''$ and $M' \not\cong_{PL} M''$ (for example, the pair obtained via Kronheimer and Mrowka's result), then the associated contracted pseudo-triangulations $K' = K(\Gamma')$ and $K'' = K(\Gamma'')$ are such that $N'(1,3) = N''(1,3) = \mathbb{D}^4$, while $N'(0,2,4)$ and $N''(0,2,4)$ both consist of $\beta_2(M') = \beta_2(M'')$ triangles with the same boundary (see Proposition 5.4). Hence, the same handle-decomposition is induced, and also with the same intersection form. However, the fact that $M' \not\cong_{PL} M''$ proves that, if $\beta_2(M') = \beta_2(M'') \geq 2$, it is not possible to identify the framed link associated to the 2-handles, despite what happens when $\beta_2(M') = \beta_2(M'') = 1$ by means of Gordon-Luecke's result.

Now, the semi-simple case of Proposition 6.1, together with results by [6], enables to extend Proposition 6.3 to the non-simply-connected case.

Proposition 6.4 *A pair of semi-simple crystallizations (of type $m \geq 1$) Γ, Γ' exists, such that*

$$|K(\Gamma)| \cong_{TOP} |K(\Gamma')| \quad \textit{but} \quad |K(\Gamma)| \not\cong_{PL} |K(\Gamma')|.$$

Proof. By Kreck (see [39]), it is known that $\mathbb{RP}^4\#K3 \cong_{TOP} \mathbb{RP}^4\#_{11}(\mathbb{S}^2 \times \mathbb{S}^2)$ but $\mathbb{RP}^4\#K3 \not\cong_{PL} \mathbb{RP}^4\#_{11}(\mathbb{S}^2 \times \mathbb{S}^2)$. Hence, to obtain a pair of semi-simple crystallizations (of type 1) proving the above statement, it is sufficient to make graph connected sums of the (unique) semi-simple crystallization of $\mathbb{RP}^4$ and either a simple crystallization of $K3$ or eleven copies of a simple crystallization of $\mathbb{S}^2 \times \mathbb{S}^2$ (see Subsection 5.2). □

Moreover, we point out that the program $\Gamma4$-*class*, performing automatic recognition of PL-homeomorphic 4-manifolds, could be an useful tool to approach open problems related to different triangulations of the same TOP 4-manifold, which are conjectured to represent the same PL 4-manifold.

For example, it is in progress its application to the open problem concerning the possible PL-equivalence of the 16- and 17-vertices triangulations of the $K3$-surface obtained in [26] and [46] respectively.

Note that similar attempts to settle the conjecture are described in [7], [8] and [9]. However, the elementary moves involved in those procedures (namely, *edge-contraction* and/or *bistellar moves*) are different from those used by our program (i.e. flips, blobs, ρ-pair switchings and dipole eliminations). Hence, it is possible that one sequence succeeds when the others fail, or viceversa, with equal computational time employed.

6.2. Additivity of regular genus and related problems

It is easy to check that the relation $\mathcal{G}(M\#M') \leq \mathcal{G}(M) + \mathcal{G}(M')$ can be stated for all PL n-manifolds by direct estimation of $\mathcal{G}(M\#M')$ on the gem $\Gamma\#\Gamma'$, when Γ, Γ' are assumed to be gems of M, M' realizing the regular genus of the represented n-manifolds. Moreover, the additivity of regular genus under connected sum has been conjectured[13], and the associated (open) problem is significant, at least in the orientable case, and especially in dimension four.

Conjecture 1 [31] *Let M_1^n, M_2^n be two closed (orientable) PL n-manifolds. Then,*

$$\mathcal{G}(M_1^n\#M_2^n) = \mathcal{G}(M_1^n) + \mathcal{G}(M_2^n).$$

In fact, it is easy to prove that the 4-dimensional case of Conjecture 1 implies the 4-dimensional Smooth Poincaré Conjecture, via the well-known Wall Theorem on homotopic 4-manifolds:

if Σ^4 is a homotopy sphere, then $\Sigma^4\#(\#_h(\mathbb{S}^2\times\mathbb{S}^2)) \cong \mathbb{S}^4\#(\#_h(\mathbb{S}^2\times\mathbb{S}^2)) \cong \#_h(\mathbb{S}^2\times\mathbb{S}^2)$, for a suitable non-negative integer h, and hence $\mathcal{G}(\Sigma^4\#(\#_h(\mathbb{S}^2\times\mathbb{S}^2))) = \mathcal{G}(\#_h(\mathbb{S}^2\times\mathbb{S}^2))$. Thus, the additivity of the regular genus would yield $\mathcal{G}(\Sigma^4) = 0$, i.e. $\Sigma^4 \cong \mathbb{S}^4$ (by Proposition 5.1, in case $n = 4$).

The following statement improves via Theorem 3.1 a double inequality concerning regular genus obtained in [38, Corollary 6.5].

Proposition 6.5 *For each closed PL 4-manifold M, with $rk(\pi_1(M)) = m$:*

$$2 - 2\mathcal{G}(M) \leq \chi(M) \leq 2 + \frac{\mathcal{G}(M)}{2} - \frac{5m}{2}.$$

In [38, Corollary 6.8], by means of the double inequality of [38, Corollary 6.5], two classes of (not necessarily orientable) PL 4-manifolds have been detected, for which additivity of regular genus holds. Now, by means of the improvement of Proposition 6.5, we can strictly enlarge the set of PL 4-manifolds for which additivity of regular genus is known to hold.

Proposition 6.6 *Let M_1, M_2 be two PL 4-manifolds, with $rk(\pi_1(M_i)) = m_i$ for each $i \in \{1, 2\}$.*

(a) If $\mathcal{G}(M_i) = 1 - \frac{\chi(M_i)}{2}$ for each $i \in \{1, 2\}$, then:

$$\mathcal{G}(M_1\#M_2) = \mathcal{G}(M_1) + \mathcal{G}(M_2) \quad \textit{and} \quad \mathcal{G}(M_1\#M_2) = 1 - \frac{\chi(M_1\#M_2)}{2}.$$

[13] Obviously, regular genus satisfies the additive property with respect to connected sum of closed 3-manifolds, via a classical result on Heegaard genus.

(b) If $\mathcal{G}(M_i) = 2\chi(M_i) + 5m_i - 4$ *for each* $i \in \{1, 2\}$, *then:*

$$\mathcal{G}(M_1 \# M_2) = \mathcal{G}(M_1) + \mathcal{G}(M_2) \quad \textit{and} \quad \mathcal{G}(M_1 \# M_2) = 2\chi(M_1 \# M_2) + 5(m_1 + m_2) - 4.$$

As pointed out in [21], the class of PL 4-manifolds involved in Proposition 6.6(a) consists of connected sums of $\mathbb{S}^3$-bundles over $\mathbb{S}^1$: in fact, by the combinatorial properties of crystallizations in dimension 4, relation $\mathcal{G}(M) = 1 - \frac{\chi(M)}{2}$ implies $\rho_\epsilon(\Gamma_{\hat{i}}) = 0$ for each $i \in \Delta_4$, and $M \cong_{PL} \#_m(\mathbb{S}^1 \otimes \mathbb{S}^3)$ directly follows from the existence of (at least) an $i \in \Delta_4$ such that $\rho_\epsilon(\Gamma_{\hat{i}}) = 0$ (see [25] for details).

On the other hand, Theorem 5.2 easily proves that the class of PL 4-manifolds involved in Proposition 6.6(b) includes all PL 4-manifolds admitting semi-simple crystallizations.

It is an open problem to completely determine this second class of PL 4-manifolds for which additivity of regular genus holds.

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