Catalogues of PL-manifolds and complexity estimations via crystallization theory

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1. Abstract

Crystallization theory is a graph-theoretical representation method for compact PL-manifolds of arbitrary dimension, with or without boundary, which makes use of a particular class of edge-coloured graphs, which are dual to coloured (pseudo-) triangulations. These graphs are usually called *gems*, i.e. *Graphs Encoding Manifolds*, or *crystallizations* if the associated triangulation has the minimal number of vertices.
One of the principal features of crystallization theory relies on the purely combinatorial nature of the representing objects, which makes them particularly suitable for computer manipulation.

The present talk focuses on up-to-date results about:

- generation of catalogues of PL-manifolds for increasing values of the vertex number of the representing graphs;
- definition and/or computation of invariants for PL-manifolds, directly from the representing graphs.

2. Cataloguing PL-manifolds via crystallization theory

Tables of crystallizations have been obtained in dimension 3, and are in progress in dimension 4: the main tool for their generation is the *code*, a numerical "string" which completely describes the combinatorial structure of a coloured graph, up to colour-isomorphisms ([12]). Afterwards suitable moves on gems, translating the PL-homeomorphism of the represented manifolds, are applied to develop a classification procedure which allows to detect crystallizations of the same manifold; this is the starting point toward the identification of the manifolds represented in the catalogues (see [7] and related C++ programs - jointly elaborated with P. Cristofori -, whose codes have been recently parallelized in order to obtain a better performance[1]).

It is worthwhile noting that in dimension 3 the above automatic partition into equivalence classes succeeds to distinguish topologically all manifolds represented by the generated catalogues. This allows to classify the 110 (resp. 16) closed prime orientable (resp. non-orientable) 3-manifolds having a coloured triangulation with at most 30 tetrahedra. The obtained results comprehend the JSJ-decomposition of all involved manifolds, together with the computation of their *Matveev complexity* and geometry: see [8] and [9] for the orientable case and [4], [5] and [1] for the non-orientable one.

[1] We expect to succeed in significantly extending crystallization catalogues, both in dimension three and in dimension four, by optimizing the code and by exploiting high-powered computers, in virtue of the Italian Supercomputing Resource Allocation (ISCRA) project *"Cataloguing PL-manifolds in dimension 3 and 4 via crystallization theory"*, supported by CINECA.

Experimental data from these catalogues also yield interesting information in order to compare Matveev complexity with the so-called *gem-complexity* of a closed 3-manifold M, which involves the minimum order of a crystallization of M ([2]).[2]

As far as dimension 4 is concerned, the generation of manifolds catalogues implies the previous generation of all gems (not necessarily crystallizations) representing 3-dimensional spheres up to a fixed order; moreover, suitable sequences of combinatorial moves realizing the PL-classification of the represented 4-manifolds have to be chosen and implemented (see [3] and [17]).

The initial segment of 4-dimensional crystallizations catalogue allows to:

- *characterize* $\mathbb{S}^4$ *(resp.* $\mathbb{CP}^2$*) (resp.* $\mathbb{S}^1 \times \mathbb{S}^3$ *and* $\mathbb{S}^1 \tilde{\times} \mathbb{S}^3$*) among closed 4-manifolds by means of gem-complexity 0 (resp. 3) (resp. 4);*
- *check that no other closed handle-free 4-manifold exists with gem-complexity* ≤ 5*;*
- *check that* $\mathbb{RP}^4$ *has gem-complexity 7 and no other closed non-orientable handle-free 4-manifold exists with gem-complexity* ≤ 9*;*[3]
- *check that* $\mathbb{S}^2 \times \mathbb{S}^2$*,* $\mathbb{CP}^2 \# \mathbb{CP}^2$ *and* $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$ *have gem-complexity 6 and any other closed orientable handle-free 4-manifold with gem-complexity* k*,* $6 \leq k \leq 9$*, is TOP-homeomorphic to one of them.*

Note that the PL-classification of the elements of our catalogue might provide interesting examples of different PL 4-manifolds triangulating the same topological 4-manifold. In fact, known properties of crystallizations, combined with the up-to-date topological classification of simply connected PL 4-manifolds (see [15], [14] and [16]), allow to prove that:
if M^4 *is a simply connected closed PL 4-manifold with gem-complexity* $k \leq 65$*, then* M^4 *is TOP-homeomorphic to either* $(\#_r \mathbb{CP}^2)\#(\#_{r'} -\mathbb{CP}^2)$ *with* $r + r' = \beta_2(M^4)$ *or* $\#_s(\mathbb{S}^2 \times \mathbb{S}^2)$ *with* $s = \frac{1}{2}\beta_2(M^4)$*.*

3. Complexity estimations

By making use of the strong connection existing in dimension 3 between gems and Heegaard diagrams, a 3-manifold invariant based on crystallization theory - called *GM-complexity* - has been introduced and proved to be an upper bound for the Matveev complexity of each compact 3-manifold (see [5], [6] and [8] for the closed case and [10] for the boundary case).

Experimental results concerning 3-manifolds admitting a crystallization with "few" vertices (namely less than 32), suggests the sharpness of this bound for all closed 3-manifolds.

[2]The *gem-complexity* of a closed $n-$manifold M^n is the non-negative integer $k(M^n) = p - 1$, $2p$ being the minimum order of a crystallization of M^n.

[3]Actually, the standard order 16 crystallization of $\mathbb{RP}^4$ turns out to be the unique non-bipartite 5-coloured graph, within the catalogue of all rigid crystallizations lacking in dipoles up to 20 vertices.

The notion of GM-complexity, combined with the widely investigated relationships between crystallization theory and other representation methods for 3-manifolds, has allowed to obtain direct estimations of the Matveev complexity for several classes of manifolds, significantly improving former results: this happens, in particular, for two-fold branched coverings of $\mathbb{S}^3$, for three-fold simple branched coverings of $\mathbb{S}^3$, and for 3-manifolds obtained by Dehn surgery on framed links in $\mathbb{S}^3$ (see [6]).

Moreover, GM-complexity has been proved to coincide with the so called *modified Heegaard complexity*, another 3-manifold invariant introduced in [13] (by making use of *generalized Heegaard diagrams*) as an approach to Matveev complexity computation: see [11].

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